A fixed-source gap above the Holevo rate for visible quantum compression

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Abstract

We construct a finite ensemble of four-qubit mixed states whose visible quantum compression rate exceeds its Holevo information even with unlimited shared classical randomness. The ensemble consists of the 510 states $(I + bP)/16$, where P ranges over the nonidentity Hermitian four-qubit Pauli operators and $b \in \{-1, 1\}$, with uniform probabilities. Its Holevo information is one bit. For every blocklength n, every quantum message dimension m, arbitrary mixed encodings and collective decoding channels, the average trace-distance error satisfies $1 - \varepsilon_n \leq m^{3/4}(53/170)^{n/2}$. The state reconstructed with the shared seed is averaged before the trace distance is evaluated. Thus the rate is at least $\frac{2}{3}\log_2(170/53) = 1.1209803\ldots$ qubits per signal, and the error tends to one exponentially below this bound. The proof combines an exact Pauli support moment with dimension bounds on the purities of partial Choi matrices. This gives a counterexample to Holevo-rate achievability for visible compression with free shared classical randomness.

1 The question and the counterexample

In visible compression, a sender knows the classical label of the quantum state to be reproduced. For an ensemble of mixed states, its Holevo information is a natural lower bound on the quantum communication rate. Whether this lower bound can always be achieved is sensitive to both the available shared resources and the approximation criterion. Winter's Conjecture 23 [1] proposes Holevo-rate visible compression with common randomness at a prescribed rate. We give an explicit counterexample even when the amount of shared classical randomness is unrestricted.

Information and compression cost were already known to differ in one-shot settings. Jain, Radhakrishnan and Sen [2] constructed random-subspace ensembles resisting compression into small subspaces. Bab Hadiashar and Nayak [3] proved one-shot incompressibility for general decoding channels and established lower bounds on combined communication and entanglement cost. Their support-projection tests are also the starting point here. The additional step is a bound for *every tensor power of one fixed finite ensemble*, with arbitrary shared classical randomness.

For comparison, Hayashi's regularized entanglement-of-purification formula describes the unassisted visible rate [4]. Nator and Pereg characterize a classical-message coordination problem with shared randomness [5]. Baghali Khanian gives visible-compression strong-converse bounds through regularized Rényi entanglement of purification in an unassisted model [6]. These resource models do not directly give the quantum-message, freely randomized converse proved below.

1.1 Coding model

All logarithms are base two. Let $\{p_x, W_x\}_{x\in\mathcal{X}}$ be a finite ensemble on $B \cong \mathbb{C}^D$, with $p_x > 0$. The labels $X_1, \ldots, X_n$ are independent with distribution p, and $W_{x^n} = W_{x_1} \otimes \cdots \otimes W_{x_n}$. Alice knows

x^n. Alice and Bob share a finite random variable S_n, independent of X^n, with arbitrary distribution q_n. Its size is unrestricted. For each seed s, Alice sends an arbitrary density operator $\tau_{x^n,s}$ on a fixed message space M_n of dimension m_n. Bob applies a completely positive, trace-preserving map $\mathcal{D}_{n,s} : \mathcal{L}(M_n) \to \mathcal{L}(B^{\otimes n})$. No shared entanglement or other communication is available.

The reconstructed state and error are

$$\widehat{W}_{x^n} = \sum_s q_n(s)\mathcal{D}_{n,s}(\tau_{x^n,s}), \tag{1}$$

$$\varepsilon_n = \frac{1}{2}\sum_{x^n} p^n(x^n)\|W_{x^n} - \widehat{W}_{x^n}\|_1. \tag{2}$$

The seed is averaged in (1), before the norm in (2). Define

$$R_{\mathrm{vis},q,\mathrm{cr}}(p,W) = \inf_{\{\text{codes}:\varepsilon_n\to 0\}} \limsup_{n\to\infty} \frac{\log m_n}{n}, \qquad \chi(p,W) = S\left(\sum_x p_x W_x\right) - \sum_x p_x S(W_x). \tag{3}$$

The universal equality $R_{\mathrm{vis},q,\mathrm{cr}} = \chi$ is the free-randomness communication part of Winter's conjecture. It is also the exact question recorded in the QIQCOP catalog [7].

1.2 Main result

Write $\mathcal{P}_4 = \{I, X, Y, Z\}^{\otimes 4}$ for the unsigned Hermitian four-qubit Pauli basis, including the identity. The signs of the individual tensor factors are fixed by the usual Pauli matrices.

Theorem 1. *Let $D = 16$. On the 510 labels (P, b) with $P \in \mathcal{P}_4 \setminus \{I\}$ and $b \in \{-1, 1\}$, take the uniform distribution and the states*

$$W_{P,b} = \frac{I + bP}{16}. \tag{4}$$

Their Holevo information is $\chi = 1$. For every positive integer n and every code in the model above, with message dimension $m = m_n$,

$$1 - \varepsilon_n \le m^{3/4}\left(\frac{53}{170}\right)^{n/2}. \tag{5}$$

Consequently,

$$R_{\mathrm{vis},q,\mathrm{cr}}(p,W) \ge R_0 := \frac{2}{3}\log\frac{170}{53} = 1.1209803210\ldots > 1 = \chi(p,W). \tag{6}$$

For every $\delta > 0$, a code with $\log m_n/n \le R_0 - \delta$ satisfies $\varepsilon_n \ge 1 - 2^{-3\delta n/4}$.

The ensemble is independent of the blocklength and the allowed error. The theorem permits mixed message states and a collective decoding channel on the entire block. Sending the original four-qubit state gives the elementary upper bound $R_{\mathrm{vis},q,\mathrm{cr}} \le 4$; the exact optimal rate is not determined. The exponential error bound applies below R_0, rather than asserting a strong converse at the unknown optimum.

2 A support-moment bound for collective decoders

Suppose a finite ensemble has support projectors Q_x with $\mathrm{Tr}(Q_x W_x) = 1$. Write $Q_{x^n} = \bigotimes_{i=1}^n Q_{x_i}$. The trace-distance variational bound gives

$$\mathrm{Tr}(Q_{x^n}\widehat{W}_{x^n}) \ge 1 - \frac{1}{2}\|W_{x^n} - \widehat{W}_{x^n}\|_1. \tag{7}$$

Thus accurate reconstruction requires high average acceptance by these support tests. We bound this acceptance uniformly over the encoder and decoder.

For a channel $\Phi : \mathcal{L}(\mathbb{C}^m) \to \mathcal{L}(\mathbb{C}^d)$, use the unnormalized Choi matrix

$$J_\Phi = \sum_{u,v=1}^{m} |u\rangle\langle v| \otimes \Phi(|u\rangle\langle v|). \tag{8}$$

It is positive semidefinite and satisfies $\mathrm{Tr}_{\mathbb{C}^d} J_\Phi = I_m$ and $\mathrm{Tr}\, J_\Phi = m$.

Lemma 2. *Every channel $\Phi : \mathcal{L}(\mathbb{C}^m) \to \mathcal{L}(\mathbb{C}^d)$ satisfies*

$$\mathrm{Tr}\, J_\Phi^2 \le \min\{m^2, md\}. \tag{9}$$

Proof. Positivity and $\mathrm{Tr}\, J_\Phi = m$ give $\mathrm{Tr}\, J_\Phi^2 \le m^2$. Let v be a normalized eigenvector associated with the largest eigenvalue λ of J_Φ. The positive-operator inequality $J_\Phi \ge \lambda |v\rangle\langle v|$ implies

$$I_m \ge \lambda \rho_M, \qquad \rho_M = \mathrm{Tr}_{\mathbb{C}^d} |v\rangle\langle v|.$$

The density matrix ρ_M has rank at most d, so its largest eigenvalue is at least $1/d$. It follows that $\lambda \le d$, and hence $\mathrm{Tr}\, J_\Phi^2 \le \lambda\, \mathrm{Tr}\, J_\Phi \le dm$. □

Let F_D denote the swap on two copies of $\mathbb{C}^D$. The next lemma is the part of the argument that controls arbitrary block decoders.

Lemma 3. *Assume*

$$\mathbb{E}_x Q_x \otimes Q_x = aI + cF_D, \qquad a, c \ge 0. \tag{10}$$

Fix any channel $\mathcal{D} : \mathcal{L}(\mathbb{C}^m) \to \mathcal{L}((\mathbb{C}^D)^{\otimes n})$, and let $T = \mathcal{D}^*$ be its Hilbert–Schmidt adjoint. For each $S \subseteq [n]$, define $\mathcal{D}_S = \mathrm{Tr}_{S^c} \circ \mathcal{D}$ and $J_S = J_{\mathcal{D}_S}$. Then

$$\mathbb{E}_{x^n} \mathrm{Tr}[T(Q_{x^n})^2] = \sum_{S \subseteq [n]} a^{n-|S|} c^{|S|}\, \mathrm{Tr}\, J_S^2, \tag{11}$$

$$\mathbb{E}_{x^n} \mathrm{Tr}[T(Q_{x^n})^2] \le m^{1+t}(a + cD^{1-t})^n \qquad (0 \le t \le 1). \tag{12}$$

Proof. Independence of the labels gives

$$\mathbb{E}_{x^n} Q_{x^n} \otimes Q_{x^n} = \sum_{S \subseteq [n]} a^{n-|S|} c^{|S|} F_S, \tag{13}$$

where F_S swaps the two copies of the sites in S and acts as the identity on all other sites. Tensor factors are grouped into the two full output blocks on both sides of this equality.

The swap trick expresses the contribution of F_S to the left-hand side of (11) as

$$\mathrm{Tr}\big[F_m (T \otimes T)(F_S)\big].$$

Choose a Hilbert–Schmidt orthonormal Hermitian basis $\{H_j\}_{j=1}^{D^{2|S|}}$ on the sites in S. Since $F_{D^{|S|}} = \sum_j H_j \otimes H_j$ and $T(H_j \otimes I_{S^c}) = \mathcal{D}_S^*(H_j)$, this contribution equals

$$\sum_j \mathrm{Tr}[\mathcal{D}_S^*(H_j)^2] = \|\mathcal{D}_S^*\|_{\mathrm{HS,super}}^2 = \|\mathcal{D}_S\|_{\mathrm{HS,super}}^2 = \mathrm{Tr}\, J_S^2. \tag{14}$$

Here the squared superoperator norm is the sum of the squared Hilbert–Schmidt norms of the images of an orthonormal operator basis. Taking the adjoint preserves this sum. In matrix units its

value is $\sum_{u,v,i,j} |[\mathcal{D}_S(|u\rangle\langle v|)]_{ij}|^2$, which is also the sum of the squared absolute values of the entries of J_S. This proves (14) and then (11). For $S = \varnothing$, $\mathcal{D}_S$ is the trace map and $J_S = I_m$, so that the contribution is m.

Each reduced channel has output dimension $D^{|S|}$. Lemma 2 therefore gives the intermediate bound

$$\mathbb{E}_{x^n} \operatorname{Tr}[T(Q_{x^n})^2] \le \sum_{k=0}^{n} \binom{n}{k} a^{n-k} c^k \min\{m^2, mD^k\}. \tag{15}$$

For positive u, v and $0 \le t \le 1$, $\min\{u, v\} \le u^t v^{1-t}$. Thus the minimum in (15) is at most $m^{1+t} D^{(1-t)k}$. The binomial theorem gives (12). □

Corollary 4. *Under (10), every code in the model of Section 1 satisfies*

$$1 - \varepsilon_n \le m^{(1+t)/2} (a + cD^{1-t})^{n/2} \qquad (0 \le t \le 1). \tag{16}$$

Proof. First fix a decoder $\mathcal{D}$ and arbitrary message states τ_{x^n}. Complete positivity implies $T(Q_{x^n}) \ge 0$, and consequently

$$\begin{aligned} \operatorname{Tr}[Q_{x^n} \mathcal{D}(\tau_{x^n})] &= \operatorname{Tr}[T(Q_{x^n}) \tau_{x^n}] \\ &\le \|T(Q_{x^n})\|_\infty \le \sqrt{\operatorname{Tr}[T(Q_{x^n})^2]}. \end{aligned}$$

Averaging over x^n and applying the scalar Cauchy–Schwarz inequality and Lemma 3 bounds the average acceptance by the right-hand side of (16).

Apply this bound separately to each decoder $\mathcal{D}_{n,s}$. Independence of S_n and X^n keeps the label distribution equal to p^n in every branch. The upper bound is independent of s, so its average also bounds $\sum_{x^n} p^n(x^n) \operatorname{Tr}[Q_{x^n} \widehat{W}_{x^n}]$. By (7), this average is at least $1 - \varepsilon_n$. The argument averages an affine acceptance probability; it does not move the seed average through a trace norm. □

3 The Pauli ensemble

We prove Theorem 1 by computing the two-copy support moment exactly. More generally, let $D = 2^q$ and let $\mathcal{P}_q$ be the unsigned Hermitian q-qubit Pauli basis. Its orthogonality is $\operatorname{Tr}(PP') = D\delta_{P,P'}$. Completeness and Hermiticity give

$$\sum_{P \in \mathcal{P}_q} P \otimes P = DF_D. \tag{17}$$

Indeed, its matrix entries obey $\sum_P P_{ij} P_{kl} = D\delta_{il}\delta_{jk}$, by applying completeness to $P_{ij}\overline{P_{lk}}$ and using $\overline{P_{lk}} = P_{kl}$.

For nonidentity P, set $Q_{P,b} = (I + bP)/2$. It is a rank-$D/2$ projector, and $W_{P,b} = 2Q_{P,b}/D$. Uniformly averaging the signs and nonidentity Paulis gives

$$\begin{aligned} \mathbb{E}_{P,b} Q_{P,b} \otimes Q_{P,b} &= \frac{1}{4}\left(I + \frac{DF_D - I}{D^2 - 1}\right) \\ &= aI + cF_D, \qquad a = \frac{D^2 - 2}{4(D^2 - 1)}, \quad c = \frac{D}{4(D^2 - 1)}. \end{aligned} \tag{18}$$

Both coefficients are positive. For $D = 16$ and $t = 1/2$,

$$a + cD^{1-t} = \frac{127}{510} + 4\frac{4}{255} = \frac{53}{170}. \tag{19}$$

Corollary 4 proves (5).

Each state in (4) is uniform on an eight-dimensional subspace, so its entropy is three bits. The ensemble average is $I/16$, of entropy four, and hence $\chi = 1$. For $\varepsilon_n < 1$, taking logarithms in (5) gives

$$\frac{\log m_n}{n} \geq \frac{2}{3} \log \frac{170}{53} + \frac{4}{3n} \log(1 - \varepsilon_n). \tag{20}$$

The last term tends to zero along any vanishing-error code sequence. This bounds its lower limit, and therefore also the upper limit used in (3). Strictness above one follows from the integer comparison

$$8 \cdot 53^2 = 22472 < 28900 = 170^2.$$

Finally, substituting $\log m_n/n \leq R_0 - \delta$ into (5) gives the exponential bound in the theorem. This completes its proof.

4 Interpretation and verification

The support constraints of this ensemble persist under tensor powers. Their second moment separates into partial swaps, and each partial swap measures the purity of a reduced decoder's Choi matrix. Its output dimension and the common message dimension impose two simultaneous purity bounds. Interpolating those bounds makes the resulting acceptance estimate decay even at one qubit of communication per signal. Independent classical randomness only mixes decoders, so it cannot evade this uniform estimate.

The theorem answers the universal achievability question negatively using one finite mixed-state source. It does not give a rate formula for general ensembles, an exact rate for this source, or a converse with shared entanglement. It uses the block trace-distance criterion (2); a criterion testing only individual output marginals is a different task.

The accompanying source package contains the frozen derivation, an analytical audit and diagnostic computations. Exact Pauli identities, finite arbitrary-channel moment comparisons and extremal channel checks were supplemented by a separately implemented exact Gaussian-rational calculation for a collective two-site channel. The latter directly enumerates 36 labels and constructs all four partial Choi matrices. Intentionally incorrect contractions and omitted terms are detected. These finite checks test algebra and implementation; the proof above establishes the arbitrary-blocklength statement.

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References

[1] A. Winter, *Compression of sources of probability distributions and density operators*, arXiv:quant-ph/0208131 (2002), Sec. VII, Conjecture 23. arXiv:quant-ph/0208131.

[2] R. Jain, J. Radhakrishnan and P. Sen, *A direct sum theorem in communication complexity via message compression*, in ICALP 2003, Lecture Notes in Computer Science **2719**, 300–315 (2003), Theorem 5. arXiv:cs/0304020.

[3] S. Bab Hadiashar and A. Nayak, *On the entanglement cost of one-shot compression*, Quantum **4**, 286 (2020), Theorems 1.1 and 3.3. doi:10.22331/q-2020-06-25-286; arXiv:1905.02110.

[4] M. Hayashi, *Optimal visible compression rate for mixed states is determined by entanglement of purification*, Phys. Rev. A **73**, 060301(R) (2006). doi:10.1103/PhysRevA.73.060301; arXiv:quant-ph/0511267.

[5] H. Nator and U. Pereg, *Coordination capacity for classical-quantum correlations*, 2024 IEEE Information Theory Workshop (2024), Corollary 3. doi:10.1109/ITW61385.2024.10807032; arXiv:2404.18297.

[6] Z. Baghali Khanian, *Strong converse bounds for compression of mixed states*, arXiv:2206.09415v2 (2025), Secs. II and IV. arXiv:2206.09415v2.

[7] QIQCOP Zoo, *Visible quantum compression with free shared randomness at the Holevo rate*, record op_1ab8b10386bddd66, revised 14 September 2026. Problem statement.