A Fourth-Level Non-Semi-Clifford Gate on Two Qudits of Every Odd Prime Dimension

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Abstract

For every odd prime p, we give an explicit gate on two p-dimensional qudits that belongs to the fourth level, but not the third level, of the Clifford hierarchy and is not semi-Clifford. The gate is a product of a cubic diagonal phase and a quadratic shear. Every Pauli conjugate is a product of an affine permutation and a phase of degree at most three, proving fourth-level membership directly for all Paulis. Exactly p Pauli classes have Pauli images, whereas a two-qudit semi-Clifford gate must preserve at least p^2 such classes. The proof includes $p = 3$ without any division by factorials. A separate double conjugation proves that the level is exactly four. This gives a negative answer to the universal two-qudit semi-Clifford question recorded as QIQCOP `op_cdd1f718ccfbccf6`. Exact computations provide reproducible checks of the smallest instance; the arbitrary-prime claim is proved algebraically.

1 Question and main result

Semi-Clifford gates admit a Clifford–diagonal–Clifford decomposition. Their role in efficient gate teleportation motivates the question of when they exhaust a level of the Clifford hierarchy. De Silva proposed the all-level statement for one or two qudits [1, Sec. V.C and Sec. VI]. Chen and de Silva proved the two-qudit statement at level three in every odd prime dimension [2, Theorem 1]. De Silva and Lautsch proved the all-level one-qudit statement and proposed a two-qudit normal form that would imply the corresponding semi-Clifford property [3, Theorem 6 and Sec. 5, Conjecture 1].

The remaining universal two-qudit statement, for odd primes and levels $k \geq 4$, is recorded in QIQCOP [5]. We give an explicit negative answer, without embedding a multiqubit construction in an encoded subspace. The proof uses only elementary polynomial identities and the defining conjugation action of the hierarchy.

Fix an odd prime p, put $\omega = \exp(2\pi i/p)$, and use the basis $\{|x, y\rangle : x, y \in \mathbb{F}_p\}$ of $(\mathbb{C}^p)^{\otimes 2}$. All coordinates and phase exponents below are evaluated in $\mathbb{F}_p$. Write $X|x\rangle = |x + 1\rangle$ and $Z|x\rangle = \omega^x|x\rangle$, and let $\mathcal{P}_{2,p}$ be generated by the single-site X_i, Z_i and arbitrary global phases. Define

$$\mathcal{C}_1(2, p) = \mathcal{P}_{2,p}, \qquad \mathcal{C}_{j+1}(2, p) = \{V : VPV^\dagger \in \mathcal{C}_j(2, p) \text{ for every } P \in \mathcal{P}_{2,p}\}. \tag{1.1}$$

The unitaries in $\mathcal{C}_2(2, p)$ are the Clifford group. A gate is *semi-Clifford* if it has the form $C_L E C_R$, where C_L, C_R are Clifford and E is diagonal. The diagonal factor is not restricted in the nonmembership argument.

Theorem 1. *For every odd prime p, define*

$$S|x,y\rangle = |x, y+x^2\rangle, \qquad D|x,y\rangle = \omega^{xy^2}|x,y\rangle, \qquad U = DS. \tag{1.2}$$

Then

$$U \in \mathcal{C}_4(2,p) \setminus \mathcal{C}_3(2,p), \qquad U \notin \mathrm{SC}(2,p). \tag{1.3}$$

More precisely, for $P(a,b,c,d) = X_1^a X_2^b Z_1^c Z_2^d$, the conjugate $UP(a,b,c,d)U^\dagger$ is a Pauli up to phase if and only if $a = b = d = 0$. Consequently, $\mathcal{C}_k(2,p) \not\subseteq \mathrm{SC}(2,p)$ for every $k \geq 4$.

The shear S is bijective, with inverse $(x,y) \mapsto (x, y-x^2)$, and D is diagonal unitary. Thus U is a well-defined unitary. To make operator order explicit, its full basis action is

$$U|x,y\rangle = \omega^{x(y+x^2)^2}|x, y+x^2\rangle. \tag{1.4}$$

2 Elementary hierarchy facts

We give the short ingredients needed for membership, including their characteristic-three scope. For $t = (x,y) \in \mathbb{F}_p^2$, use the Pauli representatives $P(v,w)|t\rangle = \omega^{w\cdot t}|t+v\rangle$.

Lemma 2. *Let q be a polynomial in two variables over $\mathbb{F}_p$, and let $D_q|t\rangle = \omega^{q(t)}|t\rangle$. If the total degree of q is at most two, D_q is Clifford. If it is at most three, $D_q \in \mathcal{C}_3(2,p)$.*

Proof. Direct conjugation gives

$$D_q P(v,w) D_q^\dagger |t\rangle = \omega^{w\cdot t + q(t+v) - q(t)}|t+v\rangle. \tag{2.1}$$

A finite difference lowers total polynomial degree by at least one. For degree at most two, the phase in (2.1) is affine linear, so the image is Pauli up to phase. For degree at most three, the phase has degree at most two, so the image is a product of a translation Pauli and a diagonal Clifford, hence is Clifford. This proves both assertions for every Pauli, not only the generators.

The degree statement holds in every characteristic: cancellation may lower the degree further. In particular, no division by 3 or 3! is used when $p = 3$, and no uniqueness of cubic polynomial representations as functions is required. □

Lemma 3. *If $A(t) = Lt + r$ is an invertible affine map of $\mathbb{F}_p^2$, then its permutation gate $T_A|t\rangle = |A(t)\rangle$ is Clifford. Left multiplication by a Clifford preserves $\mathcal{C}_3(2,p)$.*

Proof. The conjugate $T_A P(v,w) T_A^\dagger$ has translation Lv and phase $w \cdot L^{-1}(t-r)$, which is affine linear. For the second claim, if $V \in \mathcal{C}_3$ and C is Clifford, then $(CV)P(CV)^\dagger = C(VPV^\dagger)C^\dagger$ is Clifford for every Pauli P, since the Clifford operators form a group. □

The second assertion does not require $\mathcal{C}_3$ to be a group. We do not infer membership of a product of two arbitrary third-level gates.

3 All-Pauli conjugation and exact level

Let $f(x,y) = xy^2$ and fix arbitrary $a,b,c,d \in \mathbb{F}_p$. Applying $U^\dagger$, $P(a,b,c,d)$, and U in that order gives

$$UP(a,b,c,d)U^\dagger|x,y\rangle = \omega^{h(x,y)}|A_{a,b}(x,y)\rangle, \tag{3.1}$$

$$A_{a,b}(x,y) = (x+a, y+b+2ax+a^2), \tag{3.2}$$

$$h(x,y) = cx + d(y-x^2) + f(A_{a,b}(x,y)) - f(x,y). \tag{3.3}$$

Indeed, $U^\dagger = S^\dagger D^\dagger$ first supplies the phase $-f(x,y)$ and sends y to $y-x^2$. The Pauli contributes $cx + d(y-x^2)$ and the translation (a,b). The final S produces $A_{a,b}$ and the final D contributes $f(A_{a,b})$.

The linear part of $A_{a,b}$ is $\begin{pmatrix} 1 & 0 \\ 2a & 1 \end{pmatrix}$, of determinant one, and the degree of h is at most three. Hence (3.1) is the gate $T_{A_{a,b}} D_h$, which belongs to $\mathcal{C}_3$ by Lemmas 2 and 3. Since all Pauli labels were arbitrary and global phases do not affect membership, this proves $U \in \mathcal{C}_4(2,p)$ directly from (1.1).

To establish the exact level without appealing to a classification of $\mathcal{C}_3$, put $V = UX_1U^\dagger$. In (3.2)–(3.3), this is the choice $(a,b,c,d) = (1,0,0,0)$. Thus $A(x,y) = (x+1, y+2x+1)$ and $h(x,y) = (x+1)(y+2x+1)^2 - xy^2$. A second conjugation gives

$$VX_2V^\dagger |x,y\rangle = \omega^{4x^2-6x+2y+3} |x, y+1\rangle. \tag{3.4}$$

For example, the phase is obtained by evaluating $h(t+e_2) - h(t)$ at $t = A^{-1}(x,y) = (x-1, y-2x+1)$. Its second forward difference in x is 8, which is nonzero in every odd prime field. A Pauli phase is affine linear and has vanishing second differences, so (3.4) is not Pauli up to phase. Therefore V is not Clifford and $U \notin \mathcal{C}_3(2,p)$.

4 The obstruction to semi-Clifford structure

We now determine all Pauli-to-Pauli conjugations of U. If $a \neq 0$, the second-coordinate increment of $A_{a,b}$ is $b + 2ax + a^2$, which depends nontrivially on x because p is odd. A Pauli always has a translation as its basis permutation. Thus (3.1) cannot be Pauli when $a \neq 0$.

When $a = 0$, the permutation is a translation and the phase simplifies to

$$h(x,y) = (c+b^2)x + dy - dx^2 + 2bxy. \tag{4.1}$$

Write $\Delta_x h = h(x+1,y) - h(x,y)$ and define Δ_y similarly. The phase of a Pauli, modulo a constant scalar phase, is affine linear, whereas (4.1) satisfies

$$\Delta_x \Delta_y h = 2b, \qquad \Delta_x^2 h = -2d. \tag{4.2}$$

Since 2 is invertible in $\mathbb{F}_p$, Pauli membership forces $b = d = 0$. Conversely, when $a = b = d = 0$, the conjugate is exactly Z_1^c. Therefore the Pauli labels with Pauli images are precisely

$$\{(0,0,c,0) : c \in \mathbb{F}_p\}, \tag{4.3}$$

a set of p classes modulo global phases.

Suppose, for a contradiction, that $U = C_L E C_R$ with C_L, C_R Clifford and E diagonal. The subgroup

$$H = C_R^\dagger \langle Z_1, Z_2 \rangle C_R \tag{4.4}$$

has p^2 distinct Pauli classes. Since E commutes with Z_1 and Z_2,

$$UHU^\dagger = C_L \langle Z_1, Z_2 \rangle C_L^\dagger \tag{4.5}$$

consists of Paulis. This contradicts (4.3). The argument permits arbitrary Clifford factors, including nondiagonal and nonmonomial ones.

Finally, the hierarchy is nested. Paulis normalize the Pauli group, so $\mathcal{C}_1 \subseteq \mathcal{C}_2$; the remaining inclusions follow by induction from (1.1). Thus the same U belongs to every $\mathcal{C}_k(2,p)$ for $k \geq 4$ while remaining non-semi-Clifford. This completes the proof of Theorem 1.

5 Smallest instance, verification, and scope

For $p = 3$, order the basis lexicographically as $(0,0), (0,1), (0,2), (1,0), \ldots, (2,2)$, indexed from zero to eight. Equation (1.4) gives the following complete monomial matrix specification:

Input column j	0	1	2	3	4	5	6	7	8
Output row	0	1	2	4	5	3	7	8	6
Exponent of ω	0	0	0	1	1	0	2	2	0

All other entries are zero. Only I, Z_1, Z_1^2 have Pauli images, which is already insufficient for semi-Cliffordness: nine classes would be necessary. This single two-qutrit instance is sufficient to negate the universal archived question. The general-prime proof is stronger than that minimal requirement.

Exact checks. Two separate implementations reproduced the qutrit result: a monomial permutation/phase implementation and a dense 9×9 implementation over $\mathbb{Z}[\omega]$, with $\omega^2 + \omega + 1 = 0$. The latter represents entries by integer pairs and uses no floating-point tolerance. A strict recursive mode checks all 81 Pauli classes at each successful hierarchy node. It confirms $U \in \mathcal{C}_4 \setminus \mathcal{C}_3$, the three preserved Pauli classes, and (3.4). Direct modular computations additionally check every Pauli label and basis vector in (3.1) for $p = 3, 5, 7, 11$. These finite checks supplement, but do not replace, the proof for arbitrary odd primes.

Distinction from generalized semi-Cliffordness. The broader generalized semi-Clifford form allows an additional basis permutation between Clifford factors. Our gate has that property: $U = DS = S(S^\dagger DS)$ is a permutation times a diagonal. Thus Theorem 1 does not give a non-generalized-semi-Clifford gate. The recent construction of de Silva and Lautsch concerns a five-qubit fifth-level gate without the generalized property [4]. These are different claims.

Literature and priority. The source comparison was checked on 20 September 2026. In [4, Sec. 8, final paragraph], the authors announce forthcoming qudit counterexamples, without specifying there their number of qudits, hierarchy level, or construction. The available primary sources and targeted searches did not identify the explicit two-qudit gate above, but this is not an exhaustive novelty assessment. We make no claim of publication priority. Independent specialist review and comparison with forthcoming work remain necessary.

Limits of the result. The theorem supplies a negative answer to the archived universal semi-Clifford question; it does not classify the full two-qudit hierarchy or decide generalized semi-Cliffordness for arbitrary gates. Odd characteristic is essential to the Pauli-image obstruction. For $p = 2$, the same formulas reduce to a product of a controlled-Z and a controlled-X, both Clifford, so they give no qubit counterexample.

Reproducibility and AI assistance

The public research note and source archive are available at

https://yuxuanzhang1995.github.io/agentic-research/two-qudit-semi-clifford/.

The archive includes this LaTeX source, both exact implementations, the strict recursive audit, run outputs, environment versions, an AI-review summary, and a source-evidence manifest. Reproduction requires Python and NumPy; no research API key is required.

The proof and exposition were generated with AI assistance in an Inspect-based research workflow. An independent model reviewed the original claim, and a subsequent AI-assisted audit rederived the conjugation identities, checked the original quantifiers and primary sources, and reran exact computations. Neither process is external peer review or formal proof-assistant verification. Scientific Agent Skills [6] informed the evidence tracking and writing process. Human expert validation is not asserted by this manuscript.

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