The error prefactor for sequential discrimination of trine states

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Abstract

An observer receives N identical copies of one of three equally likely qubit states forming a trine. The minimum error probability is $4^{-N} + O(16^{-N})$ when the copies are measured one at a time with classical feedback, and $(\frac{1}{2} + o(1))4^{-N}$ when a joint measurement is allowed. The two optimal errors have the same exponent and a limiting ratio of two. For the lower bound, condition on the first measurement. The geometry of the trine bounds the sum of the squared posterior probabilities by $1/2$. Montanaro's pairwise-fidelity bound then gives error at least 4^{-N}, even if all remaining copies are measured jointly.

1 The comparison

Let J be uniform on $\{0, 1, 2\}$. Conditioned on $J = j$, the observer receives $|\psi_j\rangle^{\otimes N}$, where

$$|\psi_j\rangle = \cos(2\pi j/3)|0\rangle + \sin(2\pi j/3)|1\rangle.$$

The states form an orbit under a qubit rotation of order three. Every pair of distinct vectors has inner product $-1/2$. Write $P_{\mathrm{coll}}(N)$ for the minimum probability of a wrong guess after an arbitrary joint measurement. Write $P_{\mathrm{seq}}(N)$ for the infimum when the observer measures each copy once, discards it, and carries only classical information to the next copy. Each measurement and the final decision may use the entire outcome history and may depend on N.

For pure-state ensembles, single-copy measurements with classical feedback can attain the collective error exponent [1]. A ratio of errors asks a finer question. It retains the multiplicative constant lost on taking the logarithm and dividing by N. This is one precise reading of the asymptotic-attainment question for geometrically uniform states in the Vienna open-problem collection [2].

Theorem 1. *For the equiprobable trine and every $N \geq 2$, put $t = 4^{-N}$ and $x = (-1/2)^N$. Then*

$$t \;\leq\; P_{\mathrm{seq}}(N) \;\leq\; \frac{1 - \sqrt{1 - 4t}}{2}, \tag{1}$$

$$P_{\mathrm{coll}}(N) = \frac{2x^2}{\left(\sqrt{1 + 2x} + \sqrt{1 - x}\right)^2}. \tag{2}$$

Consequently $\lim_{N\to\infty} P_{\mathrm{seq}}(N)/P_{\mathrm{coll}}(N) = 2$.

2 What the first measurement leaves

Let n_0, n_1, n_2 be the Bloch vectors of the trine, so that $\rho_j = |\psi_j\rangle\langle\psi_j| = (I + n_j \cdot \sigma)/2$. They form an equilateral triangle on a great circle. If Π is the orthogonal projection onto their plane, then

$$\sum_j n_j = 0, \qquad \sum_j n_j n_j^{\mathsf{T}} = \tfrac{3}{2}\Pi.$$

Consider an arbitrary first-copy POVM M (positive operator-valued measure). Use the probability measure $\mu(B) = \operatorname{Tr} M(B)/2$ on its outcomes. In dimension two its density has the form $M(dy) = (I + v_y \cdot \sigma)\,\mu(dy)$, with $|v_y| \le 1$ almost everywhere. Positivity gives the Bloch-ball constraint and the definition of μ fixes the density's trace to two. Since the average state is $I/2$, μ is also the actual outcome law. Bayes' rule gives

$$p_j(y) = \frac{1 + n_j \cdot v_y}{3}, \qquad \sum_j p_j(y)^2 = \frac{1}{3} + \frac{|\Pi v_y|^2}{6} \le \frac{1}{2}. \tag{3}$$

Thus every first outcome leaves $\sum_{i<j} p_i(y)p_j(y) = (1 - \sum_j p_j(y)^2)/2 \ge 1/4$. An outcome that rules out one state attains equality.

For any measurement on pure states $|\phi_i\rangle$ with priors p_i, Montanaro's fidelity bound [3, Theorem 1] gives

$$e \;\ge\; \sum_{i<j} p_i p_j \, |\langle\phi_i|\phi_j\rangle|^2. \tag{4}$$

Here is a binary-test proof of the form we need. Merge the final guesses into two labels, i and its complement, and call the error of that decision B_i. For each $j \ne i$, use this same binary test on ϕ_i, ϕ_j with priors $p_i, 1 - p_i$. Its error is at least the binary Helstrom value $(1 - \sqrt{1 - 4u})/2 \ge u$, where $u = p_i(1 - p_i)|\langle\phi_i|\phi_j\rangle|^2$. Averaging over $j \ne i$ with weights $p_j/(1 - p_i)$ gives exactly B_i on the left. Thus $B_i \ge \sum_{j\ne i} p_i p_j |\langle\phi_i|\phi_j\rangle|^2$. Each wrong guess is counted twice in $\sum_i B_i = 2e$, which proves (4). A prior concentrated on one label gives the inequality directly.

Allow a joint measurement on the $N - 1$ remaining copies after outcome y. The conditional states are still $|\psi_j\rangle^{\otimes(N-1)}$ and every pair has squared overlap $4^{-(N-1)}$. By (3) and (4), even this joint measurement has conditional error at least $4^{-(N-1)}/4 = t$. Averaging over y proves the lower bound in (1) for every sequential protocol. The estimate holds pointwise in the posterior, including outcomes with a zero prior, so an N-dependent first measurement obeys the same bound.

3 Attaining the leading coefficient

On the first copy measure the three effects $M_j = \frac{2}{3}(I - \rho_j)$. They sum to I. Outcome j has probability $1/3$, eliminates state j, and leaves the other two states equally likely. This is the antitrine measurement used in the optimal two-copy sequential protocol [4]. For two pure states, adaptive single-copy Helstrom measurements attain the collective binary minimum error for any number of copies [5]. Applying them here gives

$$U_N = \frac{1 - \sqrt{1 - 4^{-(N-1)}}}{2}.$$

This proves the upper bound in (1). In particular $U_N(1 - U_N) = t$ and $U_N \le 2t$, so that $0 \le P_{\mathrm{seq}}(N) - t \le U_N - t = U_N^2 \le 4t^2$. The protocol therefore realizes the optimal leading coefficient.

4 The collective benchmark

The square-root measurement is optimal for equiprobable geometrically uniform pure states [6]. The collective N-copy trine has also been studied directly by Chitambar, Duan and Hsieh [7] and by Wen-Hai Zhang [8]. We give the calculation to fix the factor in (2). The three product vectors have Gram matrix $G = (1-x)I + x\mathbf{1}\mathbf{1}^{\mathsf{T}}$. For $N \geq 2$ its eigenvalues $1+2x, 1-x, 1-x$ are positive. Represent the signals on their span by the columns of $S = \sqrt{G}$. Its diagonal is constant, with value $d = (\sqrt{1+2x} + 2\sqrt{1-x})/3$. Measuring in the coordinate basis succeeds with probability d^2. For an explicit optimality certificate, let $Y = dS/3$. It dominates each weighted signal $Se_j e_j^* S/3$, since after congruence by $S^{-1/2}$ the needed inequality is $dI \succeq S^{1/2} e_j e_j^* S^{1/2}$, whose right side has sole nonzero eigenvalue d. Thus $\operatorname{Tr} Y = d^2$ bounds the success probability of every POVM and matches the attained value.

The two off-diagonal amplitudes in each column of S both equal $(\sqrt{1+2x} - \sqrt{1-x})/3$. Squaring and summing them gives (2) after rationalization. Since $x^2 = t$ and $x \to 0$, $P_{\mathrm{coll}}(N)/t \to 1/2$, while (1) gives $P_{\mathrm{seq}}(N)/t \to 1$. This proves Theorem 1.

Relation to earlier comparisons. Chitambar, Duan and Hsieh [7, Sec. IV-B3] prove that $N \geq 2$ spatially separated parties cannot attain the finite-copy collective optimum, and exhibit a local protocol whose success tends to one. Our ratio quantifies the residual errors for receivers that discard each measured copy. Sentís, Martínez-Vargas and Muñoz-Tapia [9] study a different ternary criterion that allows inconclusive outcomes and requires zero error on a guess.

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