Scalar compressions and asymptotically optimal tough error models

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Abstract

For an e-dimensional complex space of errors acting on $\mathbb{C}^n$, let $c(e,n)$ be the largest code dimension guaranteed by the Knill–Laflamme conditions without further assumptions on the errors. We prove

$$\max\left\{1, \left\lfloor \frac{n+1}{e^2+1} \right\rfloor\right\} \le c(e,n) \le \left\lceil \frac{n}{e^2+1} \right\rceil \qquad (1 \le e \le n^2).$$

The two bounds differ by at most one. Explicit error spaces attain the upper bound, so their largest codes are within one dimension of the worst possible value. The lower bound follows from a scalar-compression theorem: every D-dimensional operator system in $M_n(\mathbb{C})$ has a rank-k scalar compression when $2 \le k \le n$ and $n \ge Dk - 1$. We minimize the squared traceless parts of the compressed observables. At a positive minimum, stationarity supplies two linear dependencies; these leave a common orthogonal direction along which a sum of second variations is negative. Thus $c(e,n) = n/(e^2+1) + O(1)$ uniformly in e, n, with relative asymptotic equality whenever $n/(e^2+1) \to \infty$. Some finite integer values remain undetermined.

1 The guaranteed code dimension

How large an exactly correcting subspace must exist if only the dimension of the error space is known? This is the tough-error-model problem in Krüger and Werner's collection [1, Problem 14] and in its current catalog formulation [2]. Let $\mathcal{E} \subseteq M_n(\mathbb{C})$ be a complex linear space with $\dim_{\mathbb{C}} \mathcal{E} = e$. A nonzero subspace $C \subseteq \mathbb{C}^n$ corrects $\mathcal{E}$ if its orthogonal projection P_C satisfies

$$P_C A^* B P_C = \lambda(A,B) P_C \qquad (A, B \in \mathcal{E}) \tag{1.1}$$

for suitable scalars $\lambda(A,B)$. This is the algebraic Knill–Laflamme condition [3, 4]. Define

$$\alpha(\mathcal{E}) = \max\{\dim C : C \text{ satisfies (1.1)}\}, \qquad c(e,n) = \min_{\substack{\mathcal{E} \subseteq M_n(\mathbb{C}) \\ \dim_{\mathbb{C}} \mathcal{E} = e}} \alpha(\mathcal{E}). \tag{1.2}$$

Every line satisfies (1.1), so these quantities are positive integers. The definition imposes neither $I_n \in \mathcal{E}$ nor a trace-preserving normalization on a basis of $\mathcal{E}$. In particular, e counts a dimension of an arbitrary error space, rather than the Kraus rank of a normalized channel.

Theorem 1.1. *For all integers $n \geq 1$ and $1 \leq e \leq n^2$,*

$$L(e,n) := \max\left\{1, \left\lfloor \frac{n+1}{e^2+1} \right\rfloor\right\} \leq c(e,n) \leq U(e,n) := \left\lceil \frac{n}{e^2+1} \right\rceil. \tag{1.3}$$

There is an explicit e-dimensional error space $\mathcal{E}_{e,n}$ with $\alpha(\mathcal{E}_{e,n}) = U(e,n)$. The gap $U(e,n) - L(e,n)$ is at most one.

Thus the construction is asymptotically optimal whenever the guaranteed code dimension diverges. More precisely,

$$\left| c(e,n) - \frac{n}{e^2+1} \right| < 1, \qquad \frac{c(e,n)}{n/(e^2+1)} \longrightarrow 1 \quad \text{if } \frac{n}{e^2+1} \longrightarrow \infty. \tag{1.4}$$

The error bound is absolute and allows e to vary with n. This answers the asymptotically matching-bounds version of the problem. It does not determine the exact rounding at every integer pair.

The new lower bound comes from treating all scalar-compression constraints at once. For a Hermitian basis $I_n, H_1, \ldots, H_{D-1}$ of an operator system, consider the sum of the squared traceless parts of X^*H_jX, where X is an $n \times k$ isometry. If this sum is positive at a minimum, stationarity makes the columns of $[X, H_1X, \ldots, H_{D-1}X]$ linearly dependent in at least two directions. When $n \geq Dk-1$, a vector perpendicular to all these columns remains. Rotating each column of X toward that vector gives a negative sum of second derivatives, contradicting minimality. Section 2 gives the argument without assumptions on the spectra or commutators of the observables.

For comparison, the historical estimate recorded in [1], from the Knill–Laflamme–Viola construction [4], has scale $n/[e^2(e^2+1)]$. Reimpell also studied this problem through repeated Hermitian compression and numerical optimization [5, Section 3.3.4]. In operator-system language, a scalar compression is a quantum anticlique. Weaver's general existence argument [6, Theorem 2.10] uses diagonal compression followed by a convex partition, with an ambient dimension requirement quadratic in the operator-system dimension. Allen and Kornell [7, Theorem 2.7] obtain sharper bounds in small dimensions: dimension at most $2\ell+1$ guarantees rank k when $n > 3^\ell(k-1)$, and dimension at most $2\ell+2$ when $n > 2 \cdot 3^\ell(k-1)$. In particular, their result gives $c(2,n) \geq \lceil n/9 \rceil$, whereas (1.3) has asymptotic coefficient $1/5$. These comparisons concern the stated general bounds; special low-dimensional results can have better integer thresholds than the theorem below.

The upper construction uses standard rank-one spanning vectors from Li and Tam [8, Section 2.1] and the block-rank obstruction in Weaver [6, Proposition 2.1]. Section 3 assembles these ingredients into an e-dimensional error space, including the zero label required by the unnormalized formulation. The construction and its exact largest-code calculation are included in full. Geometric and variational descriptions of scalar compression also occur in recent work on Pauli detection [9, 10]; the theorem here is a uniform existence statement for arbitrary operator systems, with no prescribed Pauli or representation structure.

2 A variational scalar-compression theorem

An *operator system* $\mathcal{V} \subseteq M_n(\mathbb{C})$ is a complex linear subspace containing I_n and satisfying $\mathcal{V}^* = \mathcal{V}$. A rank-$k$ *scalar compression* of $\mathcal{V}$ is an isometry $X : \mathbb{C}^k \longrightarrow \mathbb{C}^n$ for which X^*HX is a scalar multiple of I_k for every $H \in \mathcal{V}$. Equivalently, $P = XX^*$ is a rank-k projection with $P\mathcal{V}P = \mathbb{C}P$.

Theorem 2.1. *Let $\mathcal{V} \subseteq M_n(\mathbb{C})$ be an operator system of complex dimension D. If $2 \le k \le n$ and*

$$n \ge Dk - 1, \tag{2.1}$$

then $\mathcal{V}$ has a rank-k scalar compression. Consequently, every such system has a scalar compression of rank at least

$$\min\left\{n, \max\left\{1, \left\lfloor \frac{n+1}{D} \right\rfloor\right\}\right\}. \tag{2.2}$$

For $D = 1$, the entire space is a scalar compression.

Proof. The self-adjoint part of $\mathcal{V}$ has real dimension D and spans $\mathcal{V}$ over $\mathbb{C}$. Choose a Hermitian basis $I_n, H_1, \dots, H_{D-1}$. For an isometry $X \in M_{n\times k}(\mathbb{C})$, set

$$G_j = X^* H_j X, \qquad \mu_j = \frac{\operatorname{Tr} G_j}{k}, \qquad T_j = G_j - \mu_j I_k, \qquad F(X) = \sum_{j=1}^{D-1} \operatorname{Tr}(T_j^2). \tag{2.3}$$

Each T_j is Hermitian and traceless. Thus $F(X) \ge 0$, with equality precisely when X is a scalar compression. The function is continuous on the compact space of isometries and invariant under $X \mapsto XU$ for unitary $U \in M_k(\mathbb{C})$. Fix a global minimizer X and suppose that $F(X) > 0$.

For an arbitrary variation Z of X, differentiation gives

$$dF_X[Z] = 4 \operatorname{Re} \operatorname{Tr}\left[\left(\sum_j H_j X T_j\right)^* Z\right]. \tag{2.4}$$

Every Z satisfying $X^*Z = 0$ is tangent to the isometries at X. Since (2.4) vanishes for all such Z, the horizontal component of the gradient is zero. Hence

$$\sum_j H_j X T_j = XS, \qquad S = \sum_j G_j T_j. \tag{2.5}$$

Each G_j commutes with T_j, so S is Hermitian. Introduce the block matrices

$$M = \begin{bmatrix} X & H_1 X & \cdots & H_{D-1} X \end{bmatrix}, \qquad B = \begin{bmatrix} -S \\ T_1 \\ \vdots \\ T_{D-1} \end{bmatrix}. \tag{2.6}$$

Equation (2.5) says $MB = 0$. Since $F(X) > 0$, at least one T_j is a nonzero traceless Hermitian matrix. Such a matrix has both a positive and a negative eigenvalue, and therefore rank at least two. As a block of B, it implies $\operatorname{rank} B \ge 2$. The matrix M has Dk columns, and its kernel contains the range of B. Consequently,

$$\operatorname{rank} M \le Dk - 2 < n. \tag{2.7}$$

Choose a unit vector $y \in \mathbb{C}^n$ orthogonal to the range of M. Then

$$X^* y = 0, \qquad X^* H_j y = 0 \quad (1 \le j \le D-1), \tag{2.8}$$

where the second equality uses $H_j^* = H_j$.

Write $X = [x_1, \dots, x_k]$. For each $a \in \{1, \dots, k\}$, replace the column x_a by $\cos\theta\, x_a + \sin\theta\, y$, leaving all other columns fixed. Denote this isometry by $X_a(\theta)$ and put $F_a(\theta) = F(X_a(\theta))$. Let f_a be the ath coordinate vector of $\mathbb{C}^k$, $E_a = f_a f_a^*$, and $C_a(\theta) = I_k + (\cos\theta - 1)E_a$. Then

$$X_a(\theta) = X C_a(\theta) + \sin\theta\, y f_a^*.$$

Writing $h_j = y^* H_j y$, the orthogonality in (2.8) gives the exact identity

$$X_a(\theta)^* H_j X_a(\theta) = C_a(\theta) G_j C_a(\theta) + \sin^2\theta\, h_j E_a. \tag{2.9}$$

Its expansion at zero is

$$G_j + \theta^2 \left(h_j E_a - \tfrac{1}{2}(E_a G_j + G_j E_a) \right) + O(\theta^4). \tag{2.10}$$

There is no first-order term. Since $\operatorname{Tr}(T_j^2) = \operatorname{Tr}(G_j^2) - (\operatorname{Tr} G_j)^2/k$, differentiation yields

$$\begin{aligned} F_a''(0) &= 4 \sum_j \operatorname{Tr}\Big[T_j \left(h_j E_a - \tfrac{1}{2}(E_a G_j + G_j E_a) \right) \Big] \\ &= 4 \sum_j \left(h_j (T_j)_{aa} - (G_j T_j)_{aa} \right). \end{aligned} \tag{2.11}$$

The last step again uses $G_j T_j = T_j G_j$. Summing over the k column rotations cancels the terms containing h_j, because $\operatorname{Tr} T_j = 0$. Moreover, $\operatorname{Tr}(G_j T_j) = \operatorname{Tr}(T_j^2)$. Therefore

$$\sum_{a=1}^{k} F_a''(0) = -4 \sum_j \operatorname{Tr}(T_j^2) = -4F(X) < 0. \tag{2.12}$$

Every $F_a''(0)$ must be nonnegative at a minimum, which is a contradiction. Thus $F(X) = 0$.

A rank-one scalar compression always exists. If $D \geq 2$ and the rank in (2.2) is at least two, it satisfies (2.1) and is at most n, so the first part applies. If $D = 1$, then $\mathcal{V} = \mathbb{C} I_n$ and rank n is immediate. □

Remark 2.2. The same argument shows that any positive stationary value of F has a direction of negative second variation under (2.1). The existence proof uses compactness to select a global minimum; it does not provide a running-time guarantee for finding a correcting code.

Corollary 2.3. *Every e-dimensional complex error space in $M_n(\mathbb{C})$ has a correcting code of dimension at least $L(e, n)$ in (1.3).*

Proof. Choose a complex basis $A_1, \ldots, A_e$ of $\mathcal{E}$ and form

$$\mathcal{V}_{\mathcal{E}} = \operatorname{span}_{\mathbb{C}}\{ I_n, A_i^* A_j : 1 \leq i, j \leq e \}. \tag{2.13}$$

This is an operator system of dimension $d \leq e^2 + 1$. Let $k = \lfloor (n+1)/(e^2+1) \rfloor$. If $k \geq 2$, then $k \leq n$ and $n \geq (e^2+1)k - 1 \geq dk - 1$, so Theorem 2.1 gives a rank-k scalar compression of $\mathcal{V}_{\mathcal{E}}$. The pairwise products $A_i^* A_j$ then satisfy (1.1), and sesquilinearity extends the condition to all $A, B \in \mathcal{E}$. When $k \leq 1$, any line is a correcting code. □

3 Explicit tough error spaces

We next construct the error spaces attaining the upper bound in Theorem 1.1. The basic obstruction is the following block-rank calculation [6, Proposition 2.1].

Proposition 3.1. *Let $P_1, \dots, P_t$ be nonzero mutually orthogonal projections summing to I_n, and let $\mathcal{V} = \operatorname{span}_{\mathbb{C}}\{P_1, \dots, P_t\}$. The largest rank of a scalar compression of $\mathcal{V}$ is $\max_s \operatorname{rank} P_s$.*

Proof. If a rank-k projection P satisfies $PP_sP = \lambda_s P$ for all s, positivity gives $\lambda_s \geq 0$ and summing gives $\sum_s \lambda_s = 1$. For some s, $\lambda_s > 0$, and hence $k = \operatorname{rank}(PP_sP) \leq \operatorname{rank} P_s$. Conversely, the range of any P_s is a scalar compression of $\mathcal{V}$. □

Assume first that $1 \leq e < n$, and put $t = \min\{n, e^2 + 1\}$. Let $u_1, \dots, u_e$ be the coordinate vectors in $\mathbb{C}^e$. Make an ordered list of e^2 vectors by taking all u_i first, followed, in lexicographic order on $i < j$, by

$$u_i + u_j, \qquad u_i + \mathrm{i}u_j. \tag{3.1}$$

These are the coordinate and pair vectors in the rank-one spanning construction of Li and Tam [8, Section 2.1], without normalizing their lengths. Let $v_0 = 0$, and let $v_1, \dots, v_{t-1}$ be the first $t - 1$ vectors of the list. Because $t - 1 \geq e$, all coordinate vectors are included.

Decompose $\mathbb{C}^n$ into t nonzero coordinate blocks with dimensions differing by at most one, and denote their projections by $P_0, \dots, P_{t-1}$. Explicitly, writing $n = qt + r$ with $0 \leq r < t$, take r blocks of dimension $q + 1$ and $t - r$ blocks of dimension q. Define diagonal errors

$$A_a = \sum_{s=0}^{t-1} (v_s)_a P_s \quad (1 \leq a \leq e), \qquad \mathcal{E}_{e,n} = \operatorname{span}_{\mathbb{C}}\{A_1, \dots, A_e\}. \tag{3.2}$$

On the block labelled by u_i, A_a equals $\delta_{ai} I$. Thus the A_a are linearly independent and $\dim \mathcal{E}_{e,n} = e$.

To identify their product space, write D_i for the projection of the block labelled by u_i. For $i < j$, let R_{ij} and S_{ij} be the projections of the blocks labelled by $u_i + u_j$ and $u_i + \mathrm{i}u_j$, respectively; a missing block means the zero projection. The labels give

$$A_i^* A_j = R_{ij} + \mathrm{i}S_{ij}, \qquad R_{ij} = \frac{A_i^* A_j + A_j^* A_i}{2}, \qquad S_{ij} = \frac{A_i^* A_j - A_j^* A_i}{2\mathrm{i}}. \tag{3.3}$$

They also give

$$D_i = A_i^* A_i - \sum_{\substack{p<q \\ i \in \{p,q\}}} (R_{pq} + S_{pq}), \tag{3.4}$$

$$P_0 = I_n - \sum_i D_i - \sum_{i<j} (R_{ij} + S_{ij}). \tag{3.5}$$

Every block projection therefore belongs to $\mathcal{V}_{\mathcal{E}_{e,n}}$. Conversely, the errors and all their products are constant on each block. We have proved

$$\mathcal{V}_{\mathcal{E}_{e,n}} = \operatorname{span}_{\mathbb{C}}\{P_0, \dots, P_{t-1}\}. \tag{3.6}$$

Proposition 3.1 now gives

$$\alpha(\mathcal{E}_{e,n}) = \left\lceil \frac{n}{t} \right\rceil = \left\lceil \frac{n}{e^2 + 1} \right\rceil. \tag{3.7}$$

The last equality holds because $t = e^2 + 1$ when $n \geq e^2 + 1$, and both ceilings equal one otherwise. The rank argument applies to every subspace, including codes with support across several blocks.

For $e \geq n$, let $\mathcal{E}_{e,n}$ contain all n diagonal matrix units and extend them by off-diagonal matrix units, in lexicographic order, to a linearly independent set of e errors. Since $\mathcal{V}_{\mathcal{E}_{e,n}}$ contains the n rank-one coordinate projections, the same rank argument forces every code to have dimension one.

Here $\lceil n/(e^2+1)\rceil = 1$. This covers the remaining parameters, including $n = e = 1$, and proves the upper bound in Theorem 1.1.

The smallest two-error example displaying all five labels is

$$A_1 = \mathrm{diag}(0, 1, 0, 1, 1), \qquad A_2 = \mathrm{diag}(0, 0, 1, 1, \mathrm{i}). \tag{3.8}$$

Its product system, with the identity adjoined as in (2.13), is the full diagonal algebra in $M_5(\mathbb{C})$, so its largest code has dimension one. Replacing the five coordinates by balanced blocks gives largest code dimension $\lceil n/5\rceil$ for every $n \geq 5$.

4 Asymptotics and the remaining integer gap

It remains to check the one-dimension gap in Theorem 1.1. Write $D = e^2 + 1$ and $n = qD + r$, with $0 \leq r < D$. If $r = 0$ or $r = D - 1$, the two sides of (1.3) agree. For $1 \leq r \leq D - 2$ and $q \geq 1$, they are q and $q + 1$. If $q = 0$, both bounds are one. This proves the stated gap and (1.4). It also gives, uniformly in all admissible parameters,

$$c(e, n) = \Theta\left(\max\left\{1, \frac{n}{e^2+1}\right\}\right). \tag{4.1}$$

For example, absolute comparison constants $1/2$ and 2 follow directly from (1.3).

The bounds recover $c(1, n) = \lceil n/2\rceil$, also a consequence of the Hermitian higher-rank numerical-range formula [11]. They give $c(e, n) = 1$ whenever $n \leq e^2 + 1$, and exact values at $n = q(e^2 + 1)$ and $n = q(e^2 + 1) - 1$ for every admissible positive q. For two errors, this yields $c(2, 9) = c(2, 10) = 2$ and $c(2, 14) = c(2, 15) = 3$. At $(e, n) = (2, 6)$, the present argument leaves $1 \leq c(2, 6) \leq 2$.

The corresponding integer question for arbitrary operator systems is whether every D-dimensional system has a scalar compression of rank $\lceil n/D\rceil$. This is a form of Weaver's conjecture on the lower quantum Turán numbers [6, following Theorem 2.11]. For $D = 3$, the stronger threshold $n \geq 3k - 2$ is known [7, Lemma 2.3]. Theorem 2.1 gives $n \geq Dk - 1$ for every D but does not reach the conjectured threshold $n \geq D(k - 1) + 1$ in general. The remaining rounding problem for error spaces may also benefit from their factorization as products $A_i^* A_j$, which was not used beyond the dimension bound in Corollary 2.3.

Finally, normalization changes the extremal problem. If a chosen error basis satisfies $\sum_i A_i^* A_i = I_n$, then the identity already lies in the product span and $\dim \mathcal{V}_{\mathcal{E}} \leq e^2$. The scalar-compression theorem still applies with that smaller dimension. The explicit models in Section 3, however, use a common zero block and need not satisfy this normalization. Equation (1.3) concerns the arbitrary linear error spaces in (1.2).

References

[1] O. Krüger and R. F. Werner (eds.), *Some Open Problems in Quantum Information Theory* (2005), Problem 14, arXiv:quant-ph/0504166.

[2] QIQCOP Zoo, *Tough error models*, problem record op_b1b41f737f9b4aeb, version dated 4 September 2026; accessed 13 November 2026.

[3] E. Knill and R. Laflamme, Theory of quantum error-correcting codes, *Physical Review A* **55**, 900–911 (1997), doi:10.1103/PhysRevA.55.900.

[4] E. Knill, R. Laflamme, and L. Viola, Theory of quantum error correction for general noise, *Physical Review Letters* **84**, 2525–2528 (2000), doi:10.1103/PhysRevLett.84.2525; arXiv:quant-ph/9908066.

[5] M. Reimpell, *Quantum Information and Convex Optimization*, doctoral dissertation, Technische Universität Braunschweig (2008), Section 3.3.4, archived dissertation.

[6] N. Weaver, The "quantum" Turán problem for operator systems, *Pacific Journal of Mathematics* **301**, 335–349 (2019), doi:10.2140/pjm.2019.301.335; arXiv:1802.07394.

[7] A. Allen and A. Kornell, The quantum Ramsey numbers $QR(2, k)$, *Linear and Multilinear Algebra* **73**, 3959–3964 (2025), doi:10.1080/03081087.2025.2578529; arXiv:2507.04128v2.

[8] C.-K. Li and B.-S. Tam, A note on extreme correlation matrices, *SIAM Journal on Matrix Analysis and Applications* **15**, 903–908 (1994), doi:10.1137/S0895479892240683.

[9] A. Gupta, B. Sun, X. He, and B. Zeng, *Variance Geometry of Exact Pauli-Detecting Codes: Continuous Landscapes Beyond Stabilizers* (2026), arXiv:2604.21800v1.

[10] B. Sun, N. Cao, Y. T. Poon, and B. Zeng, *Geometry of Knill–Laflamme Coefficients for Pauli Error Detection* (2026), arXiv:2610.01992v1.

[11] M.-D. Choi, D. W. Kribs, and K. Życzkowski, Higher-rank numerical ranges and compression problems, *Linear Algebra and its Applications* **418**, 828–839 (2006), doi:10.1016/j.laa.2006.03.019; arXiv:math/0511278.