Thermal mutual information of order $\beta^{1/4}$ in one dimension

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Abstract

We construct one-dimensional Hamiltonians with fixed local dimension, nearest-neighbor interactions and uniformly bounded local terms whose Gibbs states have middle-cut mutual information at least $c\beta^{1/4}$ along an explicit low-temperature sequence. The construction modifies the two-ring model of Gottesman and Hastings to allow a finite density of mobile holes. A synchronized boundary interaction preserves the ordering of singlet pairs. Endpoint transport estimates and an overlapping-singlet constraint give an N^{-3} energy separation from every unwanted spin sector. A weak local potential selects a central density band. A pointwise ground-state transform then controls all charge sectors and yields the bound for the full Gibbs state at $\beta = 2^{44}N^4$. The result improves the exponent $1/5$ in the previous thermal lower-bound construction. The accompanying source package contains the proof review and exact finite controls; external expert review and comprehensive priority assessment remain outstanding.

1 Question and result

The mutual information across a cut measures correlations that survive in a thermal state, including both classical and quantum contributions. For a one-dimensional Hamiltonian with bounded local interactions, an area law allows this quantity to depend on inverse temperature even when it is independent of the total chain length. How large can that temperature dependence be?

Kuwahara, Alhambra and Anshu obtain a thermal lower-bound example with growth of order $(\beta/\log\beta)^{1/5}$ [1, Sec. III.2]. Their construction uses the highly entangled ground state of the Gottesman–Hastings model [2]. The ground-state entanglement–gap relation in that model has exponent $1/4$, but transferring it to a Gibbs state also requires suppressing thermal excitations. These are distinct exponents. Our construction improves the thermal exponent by keeping a finite density of holes and controlling the unwanted sectors together.

Throughout, logarithms and entropies are natural. Write $I(L:R)_\rho = S(\rho_L) + S(\rho_R) - S(\rho)$.

Theorem 1. *For every $N \geq 2^{30}$ divisible by four, there is a Hamiltonian H_N on an open chain of N sites of dimension nine, with nearest-neighbor interactions and a uniformly bounded number of uniformly bounded terms at each site and edge, such that, for $\beta_N = 2^{44}N^4$,*

$$I(L:R)_{e^{-\beta_N H_N}/\operatorname{Tr} e^{-\beta_N H_N}} \geq \frac{\log 2}{8} N = \frac{\log 2}{8 \cdot 2^{11}} \beta_N^{1/4}. \tag{1}$$

Here $L:R$ is the middle cut of the chain.

The local coefficients may depend on N while remaining uniformly bounded. The theorem supplies a sequence of Hamiltonians and temperatures; it does not require a single translation-invariant interaction or a bound at every real temperature. Its constants are deliberately conservative.

The main difficulty is to retain the singlet structure in the unconstrained thermal ensemble. Fixing one charge sector would leave this difficulty unresolved. We instead prove an energy separation from the complement of a reducing subspace that contains many charge sectors and all their charge excitations. Local maps extract the same extensive set of singlets from every state in that subspace.

2 A local Hamiltonian with synchronized holes

Begin with two rings, L and R, each with N qutrits. The basis is $\{x, 0, 1\}$; x is a hole and $0, 1$ carry a qubit. Put $q_i = |x\rangle\langle x|_i$. On each ring include

$$h_i = \sum_{a=0,1} (|xa\rangle - |ax\rangle)(\langle xa| - \langle ax|)_{i,i+1}, \qquad 1 \le i < N. \tag{2}$$

At the four endpoints use

$$\begin{aligned} h_B &= \sum_{a,b=0,1} |v_{ab}\rangle\langle v_{ab}|, \\ v_{ab} &= |xa\rangle_{L_N,L_1}|xb\rangle_{R_N,R_1} - |ax\rangle_{L_N,L_1}|bx\rangle_{R_N,R_1}. \end{aligned} \tag{3}$$

The contact term is

$$h_P = (I - q_{L_1})(I - q_{R_1}) - |s\rangle\langle s|, \qquad |s\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}. \tag{4}$$

Finally, with $\lambda = (64N^4)^{-1}$ and cyclic indices within each ring, set

$$V = \lambda \sum_{a=L,R} \left(\sum_{i=1}^{N} q_{a,i} q_{a,i+1} - \sum_{i=1}^{N} q_{a,i} \right). \tag{5}$$

The Hamiltonian is the sum of these terms. The hopping and boundary terms have norm two, the contact term has norm one, and every potential coefficient has magnitude at most $1/64$.

Equation (3) requires a genuine particle–hole exchange on both rings. In particular it vanishes when both endpoints of either ring are holes. This differs from inserting many holes into the boundary term of [2, Eq. (7)], which can then move one ring while the other has two endpoint holes. Such a motion would destroy the alignment used below.

We use the folding geometry of [2, Sec. 2]. Pair sites $i, N+1-i$ within each ring into dimension-nine sites, obtaining two half-chains of length $N/2$, and join their endpoint pairs at the middle. Every ring edge becomes an on-site or nearest-neighbor interaction. Both (3) and (4) act on the two middle sites. The number of terms per folded edge is bounded, and the middle cut is precisely the old $L : R$ split.

3 Charges, ordered spins and endpoint transport

Fix particle counts m, k and hole counts $h = N - m, j = N - k$. Let

$$\Omega = \Omega_{N,h} \times \Omega_{N,j}, \qquad \mathcal{V}_{m,k} = (\mathbb{C}^2)^{\otimes m} \otimes (\mathbb{C}^2)^{\otimes k}, \tag{6}$$

where $\Omega_{N,h}$ consists of the h-subsets of ring sites. Sorting the spins by their occupied sites gives a unitary identification of the sector with $\ell^2(\Omega) \otimes \mathcal{V}_{m,k}$. We use the uniform probability measure π on Ω, with the corresponding constant normalization of amplitudes.

Interior hopping is $L_0 = L_L + L_R$, the sum of the two open-path exclusion Laplacians, tensored with spin identity. Let D_+ be the charge event with a hole at N and a particle at 1 on both rings; D_- reverses these occupations. The simultaneous endpoint exchange T bijects D_+ and D_-. If C_m moves the first ordered ket spin to the end, put $R = C_m^* \otimes C_k^*$. The boundary and contact forms are exactly

$$\mathcal{E}_B(f) = \mathbb{E}_\pi[1_{D_+}\|f(x) - Rf(Tx)\|^2], \tag{7}$$

$$\mathcal{E}_P(f) = \mathbb{E}_\pi[1_C\|Pf(x)\|^2], \tag{8}$$

where C means that both sites 1 are occupied, and $P = I - |s\rangle\langle s|$ on the first ordered pair. The adjoints in R compare a source amplitude with the inverse-transported target amplitude. These are identities on the whole sector, with no restriction on coherent spin superpositions.

Replace R by I in the boundary form to obtain a scalar Laplacian L. It is L_0 plus positive undirected edges and has constant ground vector. The exclusion-process gap equals the path random-walk gap [3]:

$$\delta_N = 2(1 - \cos(\pi/N)) \geq 4/N^2. \tag{9}$$

One-configuration factors are omitted; if both factors are trivial, their scalar operator is evaluated directly. The gap lower bound also follows from the exclusion quotient of the interchange process; the reverse bound uses a one-particle eigenfunction summed over occupied sites.

Write $\bar{f} = \mathbb{E}_\pi f$ and

$$\mathcal{E}_0(f) = \langle f, L_0 f\rangle_\pi = \frac{1}{2}\sum_{e \text{ adjacent}} \mathbb{E}_\pi\|f(x) - f(s_e x)\|^2. \tag{10}$$

Lemma 2. *For $F = 1_C, 1_{D_+}$ or 1_{D_-} and arbitrary vector-valued f,*

$$\|\mathbb{E}_\pi[Ff] - (\mathbb{E}_\pi F)\bar{f}\|^2 \leq 128N\mathcal{E}_0(f). \tag{11}$$

Proof. Independently in each ring, send the at most two sites inspected by F to a uniformly chosen ordered tuple of distinct sites. The map extends using at most two transpositions per ring. Averaging the transported indicator gives the constant $\mathbb{E}_\pi F$, since the number of holes is fixed.

A transposition of sites u, v is a word of $2|u - v| - 1$ adjacent swaps and uses each path edge at most twice. The combined word has length at most $8N$ and edge multiplicity at most eight. Telescoping, Cauchy–Schwarz and invariance of uniform π under every prefix permutation give

$$\mathbb{E}_\pi\|f - f\circ\sigma\|^2 \leq 128N\mathcal{E}_0(f).$$

This acts only on occupation configurations: a swap of equal occupations is the identity, and the ordered spin coordinates are fixed. Changing variables expresses the covariance in (11) as the average of $\mathbb{E}_\pi[F(f - f\circ\sigma^{-1})]$. Jensen and Cauchy–Schwarz finish the proof, also for vector-valued f. □

Suppose $m, k, h, j \in [N/4, 3N/4]$. The event probabilities satisfy

$$a = \pi(D_+) = \pi(D_-) = \frac{hmjk}{N^2(N-1)^2} \geq \frac{1}{256}, \qquad c = \pi(C) = \frac{mk}{N^2} \geq \frac{1}{16}. \tag{12}$$

Averaging the boundary difference and the contact term, and applying Lemma 2, yields

$$\|(I - R)\bar{f}\|^2 \leq 2^{26}N(\mathcal{E}_0 + \mathcal{E}_B), \tag{13}$$

$$\|P\bar{f}\|^2 \leq 2^{17}N(\mathcal{E}_0 + \mathcal{E}_P). \tag{14}$$

Indeed the respective bounds before enlarging constants are $3\mathcal{E}_B/a + 6(128N)\mathcal{E}_0/a^2$ and $2\mathcal{E}_P/c + 2(128N)\mathcal{E}_0/c^2$. Their sum is at most $2^{27}N\mathcal{E}$, where $\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_B + \mathcal{E}_P$.

4 All spin sectors

For $m = k$, use the tensor Bell basis of the m ordered pairs. The operator R rotates the Bell letters and P detects a first letter other than s. Their common kernel is

$$G_m = \mathrm{span}\{|s\rangle^{\otimes m}\}. \tag{15}$$

Every other cyclic-word orbit has period $p \le m$ and contains a position a where $P = 1$. Telescoping along that orbit gives

$$\sum_r |z_r|^2 \le 2p|z_a|^2 + 2p^2 \sum_r |z_{r+1} - z_r|^2.$$

Summing over disjoint orbits proves, for $v \perp G_m$,

$$\|v\|^2 \le 2m^2(\|(I-R)v\|^2 + \|Pv\|^2). \tag{16}$$

For unequal positive m, k, put $t = \min(m, k)$. The rotation R^t returns the shorter ring and moves the first spin of the longer ring to a different position. Thus P and $R^{-t}PR^t$ test singlets sharing one qubit but having different partners. For their singlet projections Q_1, Q_2,

$$Q_1Q_2Q_1 = \frac{1}{4}Q_1, \qquad \|Q_1 + Q_2\| = \frac{3}{2}, \qquad P + R^{-t}PR^t \ge \frac{1}{2}I. \tag{17}$$

The first identity is a direct singlet contraction; the two-projection decomposition gives the second. Tensoring unused qubits preserves them. Consequently

$$\begin{aligned} \tfrac{1}{2}\|v\|^2 &\le \|Pv\|^2 + \|PR^tv\|^2 \le 3\|Pv\|^2 + 2t^2\|(I-R)v\|^2, \\ \|v\|^2 &\le 6N^2(\|(I-R)v\|^2 + \|Pv\|^2). \end{aligned} \tag{18}$$

This covers arbitrary coherent states and avoids a least-common-multiple rotation period.

For equal counts restrict $f(x)$ pointwise to $G_m^\perp$; for unequal counts take the whole spin space. The charge Poincare inequality, followed by (13)–(18), gives

$$\mathbb{E}_\pi\|f\|^2 \le 2^{30}N^3\mathcal{E}(f). \tag{19}$$

Indeed the variance contributes at most $N^2\mathcal{E}_0/4$ and the mean contributes at most $6 \cdot 2^{27}N^3\mathcal{E}$. Projection onto G_m commutes with every Hamiltonian term, so this is a reducing-subspace bound.

5 Local density selection and a ground-state transform

The potential in a ring is minus λ times the number of nonempty hole clusters, with value zero on an empty or full ring. Hence

$$-a_N \le V \le 0, \qquad a_N = \lambda N = \frac{1}{64N^3}. \tag{20}$$

Its uniform mean in a charge sector is

$$\mu_{h,j} = \lambda\big(F_N(h) + F_N(j)\big), \qquad F_N(h) = \frac{h(h-N)}{N-1} = \frac{(h-N/2)^2 - N^2/4}{N-1}. \tag{21}$$

This follows from $\mathbb{E}q_i = h/N$ and $\mathbb{E}q_iq_{i+1} = h(h-1)/(N(N-1))$, including endpoint counts.

Let $e_{h,j}$ be the lowest eigenvalue of $L+V$. The constant trial state and Temple's inequality give

$$\mu_{h,j} - \frac{1}{4096N^4} \le e_{h,j} \le \mu_{h,j}. \tag{22}$$

For a nontrivial charge space, the second eigenvalue is at least $4/N^2 - a_N$, the trial energy belongs to $[-a_N, 0]$, and the variance is at most a_N^2. The Temple denominator is at least $3/N^2$, which gives the stated conservative error. In a one-point space equality holds directly.

We need a pointwise bound on the positive normalized scalar ground vector u, with $\mathbb{E}_\pi u^2 = 1$. The graph is connected. At $t = N^3$, its heat kernel satisfies, writing $M = |\Omega| \le 4^N$,

$$\frac{1}{2M} \le (e^{-tL})_{xy} \le \frac{3}{2M}. \tag{23}$$

The spectral error from the uniform rank-one projection is at most e^{-4N}; its relative size is at most $4^N e^{-4N} < 1/2$. The one-point case is immediate. The finite-state Feynman–Kac formula, or the product formula and (20), gives entrywise

$$e^{-tL} \le e^{-t(L+V)} \le e^{ta_N} e^{-tL}.$$

Applying this to u and canceling its eigenvalue factor proves

$$\frac{\max_x u(x)}{\min_x u(x)} \le 3e^{1/64} < 4, \qquad \frac{1}{4} < u(x) < 4. \tag{24}$$

For $\psi(x) = u(x)f(x)$, the exact ground-state transform remains valid with unitary edge twists:

$$\begin{aligned}\langle \psi, (H - e_{h,j})\psi \rangle_\pi = {} & \sum_{\{x,y\}} \pi(x)u(x)u(y)\|f(x) - U_{xy}f(y)\|^2 \\ & + \mathbb{E}_\pi[u(x)^2 1_C \|Pf(x)\|^2].\end{aligned} \tag{25}$$

The edge unitaries are I internally and R, R^* at the synchronized boundary. Expanding both sides and using $(L+V)u = e_{h,j}u$ cancels the scalar diagonal terms. Combining (24), (25) and (19) gives, on every central unwanted spin sector,

$$H \ge e_{h,j} + 2^{-38}N^{-3}. \tag{26}$$

The norm comparison costs at most 16^2. Subtracting $\|V\|$ directly from the earlier spin gap would not suffice.

In any charge sector, vector-valued diamagnetism also gives $H \ge e_{h,j}$: replace every amplitude by its norm and discard the nonnegative contact term. This includes sectors with no particles or no holes.

6 The full thermal ensemble

Let E_* be the smallest scalar ground energy among central equal-count sectors. The half-filled trial gives $E_* \le \mu_* := 2\lambda F_N(N/2)$. Equation (21) is minimized at half filling, so

$$e_{h,j} \ge E_* - \frac{1}{4096N^4} \tag{27}$$

for every sector. Together with (26), this places central unequal-count or nonsinglet sectors at least $2^{-39}N^{-3}$ above E_* for $N \geq 2^{28}$. If a hole count lies outside $[N/4, 3N/4]$, then

$$\mu_{h,j} - \mu_* \geq \lambda N/16 = \frac{1}{1024N^3}.$$

Equation (22) and diamagnetism give an even larger separation.

Define the good subspace as all charge vectors in central equal-count sectors tensored with their ordered singlet vector. It includes all charge excitations and is reducing. It contains a state at E_*. On its complement, for the declared range of N,

$$H_{\text{bad}} \geq E_* + \kappa N^{-3}, \qquad \kappa = 2^{-40}. \tag{28}$$

The complete Hilbert dimension is 3^{2N}. Thus the bad weight in the actual Gibbs state satisfies

$$\delta \leq 3^{2N} e^{-\beta\kappa/N^3} \leq e^{-(16-2\log 3)N} < e^{-13N} \tag{29}$$

at $\beta = 2^{44}N^4$.

Each ring has a local CPTP map that measures its particle number, extracts the first $r = N/4$ ordered spins and discards the remaining registers. A fixed padding rule defines the map when fewer particles are present. Every good state maps to $|s\rangle\langle s|^{\otimes r}$, including arbitrary mixtures of charge states and charge excitations. The full output is therefore

$$(1-\delta)|s\rangle\langle s|^{\otimes r} + \delta\omega$$

for some state ω. Concavity of marginal entropy gives each marginal entropy at least $(1-\delta)r\log 2$, while the joint entropy is at most $h_2(\delta) + 2\delta r \log 2$. In particular,

$$I(\text{output}) \geq 2r\log 2 - 4\delta r \log 2 - 3h_2(\delta), \tag{30}$$

where the weaker coefficient three agrees with the reviewed bound. Local data processing and (29) imply (1), with ample margin. This proves Theorem 1.

7 Relation to existing results and verification

The two-ring idea, singlet contact and folding are inherited from Gottesman–Hastings. The gap input is the exclusion-process theorem of Caputo–Liggett–Richthammer. The added construction is the strict finite-density boundary together with endpoint coercivity and local density selection. Their combination controls the full thermal ensemble and changes the thermal exponent relative to [1]. Exact finite-bond Gibbs mixtures such as [4] concern a different representation question and do not provide this lower-bound example.

The source package preserves the original statement, the reviewed proof, the present manuscript, a source comparison and executable controls. The physical Hamiltonian and charge-spin representation agree in 22,113 exact operator-column comparisons at $N = 2, 3, 4$, and 50 exact charge means agree. Four deliberately incorrect operator variants are detected. Additional numerical endpoint and spin controls are diagnostics only. New exact checks use nonconstant rational ground vectors to test the twisted transform and detect an incorrect edge weight. None of these finite calculations replaces the all-size proof.

This version received sequential internal analytical review. It has not received independent expert or proof-assistant verification. Targeted primary-source comparison supports the stated distinction from the inspected papers, without establishing comprehensive priority. The author directed the research and workflow; AI systems assisted with derivation, literature comparison, computation and drafting.

References

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