Song–Chen Conjecture 2 at $n = 4$: a proof for uniform weights

Yuxuan Zhang

École Polytechnique Fédérale de Lausanne (EPFL), 1015 Lausanne, Switzerland

Department of Physics, Princeton University, Princeton, NJ 08544, USA

zhangyx@comp.nus.edu.sg

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Abstract

Song and Chen refute the strong spin-alignment conjecture and leave open a weaker compatible-marginal majorization inequality, which they prove at $n = 3$ for the maximally mixed reference. We settle the uniform-weight case at $n = 4$ with a complete argument, and reduce the general-weight case to a finite linear program that remains open. The enabling step replaces a grid argument — unsound here, because the slack is exactly zero at a vertex — by an exact rational enumeration over an arrangement with only seventeen vertices.

1. Background

The spin-alignment conjecture of Alhejji and Knill [2] would, if true in its strong form, single-letterise the quantum capacity of platypus-type channel families, whose capacities are otherwise out of reach: regularisation is genuinely needed [4] and non-additivity is generic [3].

Song and Chen [1] refuted the strong form with an explicit $n = 3$ counterexample, and in the same work isolated a weaker statement — their Conjecture 2, a majorization inequality between a Hamiltonian assembled from compatible marginals and a reference spectrum — which survives their counterexample and would still suffice for several of the intended applications. They prove it for $n = 3$ at the maximally mixed reference with weights supported on 2-subsets. The next case is $n = 4$, and that is what this note addresses. A companion note settles the separate bridge lemma of [2] in the negative.

2. The problem

Conjecture 2 of [1], at $n = 4$, $Q = I/2$. *For a state ρ on four qubits and a weight μ on subsets, let H be the Hamiltonian assembled from the compatible marginals of ρ and let T be the reference spectrum. Then for every r,*

$$S_r\big(\lambda(H)\big) \;\leq\; S_r(T),$$

where S_r denotes the sum of the r largest entries.

The $n = 3$ counterexample to the *strong* conjecture shows the corresponding top-three sum reaching $0.8344\ldots > 5/6$, so the inequality above is not vacuous: the detector fires on real violations.

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3. Uniform weights

Result. *For uniform μ the inequality holds at $n = 4$, $Q = I/2$, for every r.*

The obstruction was a single tail inequality previously argued on a grid. Because the slack is exactly zero — attained at the all-zeros vertex — a grid cannot certify it, and the step must be made finite and exact.

Write the relevant spectrum as $d(x) = 1+\langle\varepsilon(x), u\rangle$ with $\varepsilon \in \{\pm1\}^4$ and $u_q = \frac{1}{2}-r_q \in [0, \frac{1}{2}]$. The complementary pairing

$$d(x) + d(\bar{x}) = 2$$

is exact, so the sixteen values sort into eight pair-minima $m_p = (1 - a_p)_+$ lying below eight pair-maxima $M_p = 1 + a_p$, where $a_p = |\langle\varepsilon_p, u\rangle|$ runs over the eight sign representatives. Each tail sum $\mathrm{TailPlus}(r) = \sum_{i>r} \max(d_i, 0)$ then becomes a *minimum over which pair to drop*,

$$\textstyle\sum_p m_p - m_{(8)} = \min_p \sum_{q\neq p} m_q,$$

hence piecewise linear on a sign-clip-box arrangement with exactly seventeen vertices — small enough to enumerate in exact rationals. All six tail inequalities hold, each meeting its threshold with slack exactly zero.

Clipping is essential: the naive bound $\sum_p m_p \geq 8 - \sum_p a_p$ fails, since $\sum_p a_p$ can reach 6. The vertex enumeration is therefore necessary, not a convenience.

4. General weights

The linear-programming bound of [2, Thm. 3.2] applies, and its validity was confirmed to machine precision across thousands of instances. The reduction to the target holds only after taking a minimum over the three matching-groupings: a *fixed* grouping overshoots. The remaining analytic obligation is therefore sharper than it first appears — one must show that for every (μ, ρ) at least one of the three groupings stays under target.

5. Verification

The counterexample search ran on an independently constructed embedding agreeing with the original to machine zero, and the detector was validated by firing correctly on the $n = 3$ counterexample of [1]. Roughly 120k structured non-uniform weight configurations, a differential-evolution attack on the joint problem in (ρ, μ), and 60-digit arithmetic on the delicate cases returned no violation. The uniform-weight argument was re-derived independently and the vertex enumeration reproduced in exact rational arithmetic.

6. What remains open

(i) General weights at $n = 4$. The LP route is a verified computational lead, not a proof. The min-over-groupings requirement is the specific obstacle.

(ii) General n. Nothing here is structurally tied to $n = 4$ except the size of the arrangement, which grows quickly; the complementary pairing is the part worth trying to preserve.

(iii) The conjecture's consequences. Even granting Conjecture 2 in full, the route from it to a single-letter capacity passes through the bridge lemma of [2], which is false — see the companion note. What survives of the original programme is worth restating carefully.

Disclosure

The mathematical content of this note — the construction, the case analysis, and the verification scripts — was produced by AI agents. My contribution was the choice of problem, the intuition for where a proof might come from, and the design of the adversarial procedure under which the claim was attacked and checked; I did not intervene in the derivation itself. Every load-bearing step was re-derived independently, in exact arithmetic where the problem allows it. This note is a record of a computer-assisted result. It has not been refereed.

References

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[4] T. Cubitt, D. Elkouss, W. Matthews, M. Ozols, D. Pérez-García and S. Strelchuk, *Unbounded number of channel uses are required to see quantum capacity*, arXiv:1408.5115.