

Geometric convergence of an exact randomized two-excitation cooling channel

Yuxuan Zhang

Institute of Physics, École Polytechnique Fédérale de Lausanne (EPFL)
CH-1015 Lausanne, Switzerland

Department of Physics, Princeton University
Princeton, New Jersey 08544, USA

Version 1
22 October 2026

Based on candidate M06-v1. Proof and primary-source comparisons received separate-context internal AI review; computational checks were replayed. No external expert review or formal verification is claimed.

Abstract

We prove an explicit trace-distance convergence bound for an exact bosonic collision model. A target oscillator initially at inverse temperature one interacts with successive fresh oscillators at the same temperature through the cubic exchange Hamiltonian $ab^{\dagger 2} + a^{\dagger}b^2$. Each interaction time is drawn independently and uniformly from $[1/100, 1/50]$. After N collisions the averaged target state has trace distance at most $500001 \exp[-N/(2 \cdot 10^{12})]$ from the Gibbs state at inverse temperature two. The proof applies on the full Fock space. Exact endpoint spectral moments give a uniform probability of downward motion, geometric stationary weights convert that probability into conductance, and time averaging bounds the negative spectrum. An explicit initial-tail estimate then treats an initial density outside the stationary L^2 space. We also establish finite energy moments and report heat and supplied work under both averaged-collision and history-conditional charging conventions. The certified rate is conservative.

1 The collision model and the result

Non-Gaussian excitation exchange can cool a bosonic target below the bath temperature. Png, Hu, and Scarani study this mechanism and its iterative behavior [1]. Their short-time population expansion and low-moment analysis do not by themselves give a trace-distance rate for an exact, fixed collision channel. The issue is especially significant in an infinite-dimensional system: convergence of mean energy or of a truncated numerical model does not establish convergence of the full state.

Here we fix the interaction and its random time law, and prove a geometric bound without truncating the evolving state. Set $\hbar = k_B = \omega_S = \omega_B = \beta_B = 1$. For a single oscillator define

$$\tau_\gamma = (1 - e^{-\gamma}) \sum_{r=0}^{\infty} e^{-\gamma r} |r\rangle\langle r|, \quad n_S = a^{\dagger}a, \quad n_B = b^{\dagger}b. \quad (1.1)$$

The target starts in τ_1 , and each bath oscillator is a fresh, independent copy of τ_1 . In the interaction picture the collision is

$$H = ab^{\dagger 2} + a^{\dagger}b^2, \quad U_s = e^{-isH}, \quad \Phi(\rho) = 100 \int_{1/100}^{1/50} \text{Tr}_B [U_s(\rho \otimes \tau_1) U_s^{\dagger}] \, ds. \quad (1.2)$$

The direct-sum definition of U_s is specified below. Independent time draws give the averaged state $\Phi^N(\tau_1)$.

Theorem 1.1. *For every integer $N \geq 0$, the exact channel (1.2) satisfies*

$$\frac{1}{2} \left\| \Phi^N(\tau_1) - \tau_2 \right\|_1 \leq C e^{-cN}, \quad C = 500001, \quad c = \frac{1}{2 \cdot 10^{12}}. \quad (1.3)$$

Writing $E_N = \text{Tr}[n_S \Phi^N(\tau_1)]$ and $E_* = \text{Tr}(n_S \tau_2)$, one also has

$$\text{Tr}[n_S^2 \Phi^N(\tau_1)] < 30, \quad |E_N - E_*| < 6000 e^{-N/(4 \cdot 10^{12})}. \quad (1.4)$$

The second-moment bound holds separately for every deterministic history of collision times in the allowed interval.

The protocol uses N fresh modes and total interaction time at most $N/50$. It has no retained bath mode, measurement, postselection, or additional local control. The interaction coefficient is one; its operator norm is unbounded. Thus the time and coefficient are resources of the stipulated sector model, not a uniform bound on Hamiltonian norm.

The proof has three distinct tasks. A uniform downward transition estimate supplies a positive spectral gap. Randomizing time also prevents spectrum near -1 , which a positive gap alone would not exclude. Finally, the initial thermal density is not square-integrable relative to the colder stationary law, so an exact tail estimate is needed before applying the spectral contraction.

2 Exact sectors and the reversible population chain

The operator $K = 2n_S + n_B$ is conserved. Its eigenspace with eigenvalue k has basis

$$|r, k - 2r\rangle, \quad 0 \leq r \leq \lfloor k/2 \rfloor.$$

In this basis H is the finite real symmetric Jacobi matrix H_k with zero diagonal and off-diagonal entries

$$(H_k)_{r+1,r} = (H_k)_{r,r+1} = \sqrt{(r+1)(k-2r)(k-2r-1)}. \quad (2.1)$$

The direct sum of these finite self-adjoint matrices defines a self-adjoint operator on its natural direct-sum domain, and $U_s = \bigoplus_{k \geq 0} e^{-isH_k}$ is unitary.

Put $q = e^{-1}$, and denote the initial and desired populations by

$$\mu_r = (1-q)q^r, \quad \pi_r = (1-q^2)q^{2r}. \quad (2.2)$$

The exact transition kernel, with the first index denoting the input, is

$$P(r, j) = (1-q) \sum_{\ell=0}^{\infty} q^\ell 100 \int_{1/100}^{1/50} \left| [e^{-isH_{2r+\ell}}]_{j,r} \right|^2 ds. \quad (2.3)$$

A matrix entry outside the indicated sector is zero. Nonnegativity and unitarity show that every row sums to one. For a diagonal input, two different target output occupations in the same sector have different bath occupations. The partial trace therefore removes their cross terms. Consequently the target remains diagonal and

$$\frac{1}{2} \left\| \Phi^N(\tau_1) - \tau_2 \right\|_1 = \text{TV}(\mu^{P^N}, \pi). \quad (2.4)$$

Lemma 2.1. *The law π is stationary, and P is reversible with respect to π . Both conclusions hold for each fixed-time kernel before averaging over s .*

Proof. The weight of $|r, k - 2r\rangle$ in $\tau_2 \otimes \tau_1$ equals

$$c_k = (1 - q^2)(1 - q)q^k,$$

independently of r . Each sector is therefore proportional to its identity and is fixed by conjugation with e^{-isH_k} . This proves stationarity. Since H_k is real symmetric, its exponential is symmetric as a matrix, so the squared transition moduli in opposite directions agree. Multiplication by the common sector weight and summation over sectors give $\pi_r P(r, j) = \pi_j P(j, r)$. The state identities follow by trace-norm convergence of the positive, trace-class sector sum. $\square$

We use the Markov operator on functions, $(Pf)(r) = \sum_j P(r, j)f(j)$, and the norm $\|f\|_{2,\pi}^2 = \sum_r \pi_r |f(r)|^2$. Detailed balance makes this operator self-adjoint. It also identifies evolution of a probability density relative to π with this same operator.

3 Uniform escape from every nonzero occupation

3.1 The scalar time average

Define

$$h(\omega) = 100 \int_{1/100}^{1/50} \cos(s\omega) ds = \int_1^2 \cos(tx) dt, \quad x = |\omega|/100. \quad (3.1)$$

At $x = 0$ its value is one; otherwise $h(\omega) = \cos(3x/2) \sin(x/2)/(x/2)$.

Lemma 3.1. *For every real ω ,*

$$h(\omega) \geq -\frac{99}{100}.$$

If $|\omega| \geq \sqrt{2}$, then

$$1 - h(\omega) \geq \frac{1}{10000}.$$

Proof. For $\sqrt{2}/100 \leq x \leq 1$, the inequality $1 - \cos y \geq y^2/3$ for $|y| \leq 2$ gives

$$1 - h(\omega) \geq \frac{x^2}{3} \int_1^2 t^2 dt = \frac{7x^2}{9} \geq \frac{7}{45000} > \frac{1}{10000}.$$

For $x \geq 1$ put $u = x/2$. On $1/2 \leq u \leq 2$ the positive function $\sin u/u$ decreases, so

$$|h(\omega)| \leq 2 \sin(1/2) \leq 1 - \frac{1}{24} + \frac{1}{1920} = \frac{1841}{1920} < \frac{24}{25}.$$

For $u \geq 2$ use $|h| \leq 1/u \leq 1/2$. These inequalities prove the second assertion on the remaining range.

For the first assertion, if $0 \leq x \leq 1/2$, then $h \geq 1 - 7x^2/6 \geq 17/24$. If $x \geq 1/2$, the same sinc argument now starts at $u = 1/4$. On $1/4 \leq u \leq 2$,

$$|h| \leq 4 \sin(1/4) \leq 1 - \frac{1}{96} + \frac{1}{30720} = \frac{30401}{30720} < \frac{99}{100},$$

and $u \geq 2$ is already covered. $\square$

3.2 Endpoint moments give downward motion

Lemma 3.2. *For every integer $r \geq 1$,*

$$P(r, \{0, \dots, r-1\}) \geq d_0, \quad d_0 = \frac{1-q}{200000}. \quad (3.2)$$

Proof. Condition on the bath being in its vacuum, whose probability is $1-q$. The input vector $|r, 0\rangle$ is an endpoint of sector $k = 2r$. Let Λ be a random eigenvalue distributed according to the spectral measure of H_{2r} at that vector. The first two squared couplings from this endpoint are $2r$ and $12(r-1)$. The second and fourth closed walks give, for every $r \geq 1$,

$$\mathbb{E}\Lambda = 0, \quad \mathbb{E}\Lambda^2 = 2r, \quad \mathbb{E}\Lambda^4 = (2r)^2 + (2r)12(r-1) = 28r^2 - 24r. \quad (3.3)$$

For independent copies Λ, Λ' and $\Omega = \Lambda - \Lambda'$,

$$\mathbb{E}\Omega^2 = 4r, \quad \mathbb{E}\Omega^4 = 80r^2 - 48r \leq 80r^2. \quad (3.4)$$

The second-moment inequality $\Pr(X \geq \theta \mathbb{E}X) \geq (1-\theta)^2(\mathbb{E}X)^2/\mathbb{E}X^2$ for a nonnegative X follows from Cauchy–Schwarz applied on that event. Taking $X = \Omega^2$ and $\theta = 1/2$ yields

$$\Pr(\Omega^2 \geq 2r) \geq \frac{1}{20}. \quad (3.5)$$

The survival amplitude is $\mathbb{E}e^{-is\Lambda}$, so its squared modulus is $\mathbb{E}\cos(s\Omega)$. Its random-time average is $\mathbb{E}h(\Omega)$. Since $1-h \geq 0$ everywhere and Lemma 3.1 applies on the displayed event, the averaged escape probability is at least $1/200000$. Every escaped basis vector in this vacuum-input sector has target occupation strictly below r . Multiplication by the vacuum weight proves (3.2). $\square$

Only two spectral moments enter this argument, but they are exact identities for every sector. No maximum occupation is introduced.

4 An absolute spectral gap on the infinite chain

Write $Q(i, j) = \pi_i P(i, j)$ and

$$\mathcal{E}(f) = \langle f, (I - P)f \rangle_\pi = \frac{1}{2} \sum_{i,j} Q(i, j) (f(i) - f(j))^2. \quad (4.1)$$

Let the conductance be

$$\phi = \inf_{0 < \pi(A) \leq 1/2} \frac{\sum_{i \in A, j \notin A} Q(i, j)}{\pi(A)}.$$

Because $\pi_0 = 1 - q^2 > 1/2$, every set in this infimum excludes zero. If $m = \min A$, then all states below m lie outside A , and

$$Q(A, A^c) \geq \pi_m P(m, \{0, \dots, m-1\}) \geq d_0 \pi_m, \quad (4.2)$$

$$\pi(A) \leq \sum_{r=m}^{\infty} \pi_r = q^{2m}. \quad (4.3)$$

Hence

$$\phi \geq d_0(1 - q^2) > \frac{12}{5 \cdot 10^6} > 2 \cdot 10^{-6}, \quad (4.4)$$

where $q < 2/5$ and $1 - q^2 > 4/5$ suffice.

For clarity, the required conductance-to-gap implication is proved here. If $g \geq 0$ and $g(0) = 0$, every positive level set of g^2 has mass below $1/2$. Coarea and reversibility give

$$\sum_{i,j} Q(i,j) |g(i)^2 - g(j)^2| \geq 2\phi \|g\|_{2,\pi}^2. \quad (4.5)$$

Cauchy–Schwarz bounds the left side by

$$\left(2\mathcal{E}(g) \sum_{i,j} Q(i,j) (g(i) + g(j))^2 \right)^{1/2} \leq \left(8\mathcal{E}(g) \|g\|_{2,\pi}^2 \right)^{1/2}.$$

Thus $\mathcal{E}(g) \geq (\phi^2/2) \|g\|_{2,\pi}^2$. Apply this to the positive and negative parts of $f - f(0)$. Their supports are disjoint, and the cross term in the Dirichlet form is nonpositive. It follows that

$$\mathcal{E}(f) \geq \frac{\phi^2}{2} \|f - f(0)\|_{2,\pi}^2 \geq \frac{\phi^2}{2} \text{Var}_\pi(f) \geq 10^{-12} \text{Var}_\pi(f). \quad (4.6)$$

Nonnegative integration justifies coarea for infinite level sets; bounded approximation and $L^2(\pi)$ continuity extend the inequality to all real L^2 functions.

A second argument controls the negative spectrum. For a real function f of finite support, set $F_k = \text{diag}(f(0), \dots, f(\lfloor k/2 \rfloor))$. In an eigenbasis of H_k , with eigenvalues $\lambda_{k,a}$, the stationary quadratic form is

$$\langle f, Pf \rangle_\pi = \sum_{k \geq 0} c_k \sum_{a,b} |(F_k)_{ab}|^2 h(\lambda_{k,a} - \lambda_{k,b}). \quad (4.7)$$

Indeed, before time averaging it is the weighted trace $\sum_k c_k \text{Tr}(F_k e^{-isH_k} F_k e^{isH_k})$. The sum of its absolute coefficient bounds equals

$$\sum_k c_k \text{Tr} F_k^2 = \sum_{r \geq 0} f(r)^2 \sum_{\ell \geq 0} c_{2r+\ell} = \|f\|_{2,\pi}^2.$$

Lemma 3.1 therefore proves $P \geq -0.99I$; density extends this bound to all $L^2(\pi)$. Combining it with (4.6), and extending to complex functions if needed, yields

$$\|P^N f\|_{2,\pi} \leq (1 - \lambda)^N \|f\|_{2,\pi}, \quad \pi(f) = 0, \quad \lambda = 10^{-12}. \quad (4.8)$$

5 The initial tail and trace-distance convergence

The initial density relative to π is $f_0(r) = q^{-r}/(1+q)$, so $\|f_0\|_{2,\pi} = \infty$. For an integer $K \geq 0$ use the subprobability measure $\mu_r^{(K)} = \mu_r \mathbf{1}_{\{r \leq K\}}$. Its omitted mass and squared density norm are exact:

$$t_K = q^{K+1}, \quad \left\| \mu^{(K)} / \pi \right\|_{2,\pi}^2 = \frac{1-q}{1+q} (K+1). \quad (5.1)$$

Center the density at the mass $1 - t_K$, apply (4.8), and use Cauchy–Schwarz to pass to $L^1(\pi)$. The omitted measure and $t_K \pi$ both have mass t_K , so their half- L^1 difference is at most t_K . This proves

$$\text{TV}(\mu P^N, \pi) \leq \frac{1}{2} \sqrt{\frac{1-q}{1+q}} (K+1) (1-\lambda)^N + q^{K+1}. \quad (5.2)$$

The full infinite kernel acts on both terms; only the initial measure was decomposed.

Take $K = N$. For $x = \lambda(N+1) > 0$, $x e^{-x} \leq e^{-1}$ and $\lambda \leq 1$ imply $(N+1) e^{-\lambda N} \leq \lambda^{-1}$. Thus

$$\sqrt{N+1} e^{-\lambda N/2} \leq \lambda^{-1/2}.$$

Using $(1 - \lambda)^N \leq e^{-\lambda N}$ and $q^{N+1} = e^{-(N+1)} \leq e^{-\lambda N/2}$ in (5.2), we obtain

$$\text{TV}(\mu P^N, \pi) \leq \left(\frac{1}{2\sqrt{\lambda}} + 1 \right) e^{-\lambda N/2},$$

which, with (2.4), is (1.3).

6 Energy moments, discarded heat, and supplied work

Trace-distance convergence alone need not control an unbounded energy observable. Here an $L^{3/2}$ estimate supplies the necessary uniform integrability. Direct summation gives

$$\|f_0\|_{3/2, \pi}^3 = \frac{(1 + \sqrt{q})^2}{1 + q} \leq 2. \quad (6.1)$$

Every fixed-time kernel preserves π and is reversible, so Jensen's inequality gives contraction on $L^{3/2}(\pi)$ for each collision and for any history. In particular $\|f_N\|_{3/2, \pi} < 2$ for the averaged and history-conditioned densities.

For $r \geq 0$, $r^6 \leq 720 \binom{r+5}{6}$. Summing the geometric generating function and applying Hölder's inequality gives

$$\mathbb{E}_{\mu P^N} r^2 \leq \|f_N\|_{3/2, \pi} \left(\mathbb{E}_{\pi} r^6 \right)^{1/3} \quad (6.2)$$

$$\leq 2 \left(\frac{720}{(1 - q^2)^6} \right)^{1/3} < 30. \quad (6.3)$$

The same proof holds for every history. Also $\mathbb{E}_{\pi} r^2 = q^2(1 + q^2)/(1 - q^2)^2 < 4$. If $\nu = \mu P^N$, then

$$|E_N - E_*| \leq \left(\sum_r r^2 |\nu_r - \pi_r| \right)^{1/2} \left(\sum_r |\nu_r - \pi_r| \right)^{1/2} \leq \sqrt{68 \text{TV}(\nu, \pi)}. \quad (6.4)$$

Together with $68C < 6000^2$, this proves (1.4).

Conservation of $2n_S + n_B$ implies that the mean energy gained by the discarded bath in collision j is

$$Q_j = 2(E_{j-1} - E_j). \quad (6.5)$$

The signed change in total bare energy is $E_{j-1} - E_j$, whose positive part is supplied mean work. These identities do not assume monotone target energy.

Proposition 6.1. *With positive work charged after averaging each collision,*

$$W_N^{\text{av}} = \sum_{j=1}^N [E_{j-1} - E_j]_+ \leq 10^{17} \quad \text{for every } N. \quad (6.6)$$

If it is charged separately conditional on each random time and the preceding history, its expected gross total satisfies

$$W_N^{\text{hist}} \leq 6N. \quad (6.7)$$

Under both conventions, the expected signed heat and net work are respectively $2(E_0 - E_N)$ and $E_0 - E_N$, with limits

$$Q_{\infty}^{\text{signed}} = \frac{2q}{1 - q^2}, \quad W_{\infty}^{\text{net}} = \frac{q}{1 - q^2}. \quad (6.8)$$

Proof. Set $d = 1/(4 \cdot 10^{12})$ and $\delta_j = E_j - E_*$. Then $[E_{j-1} - E_j]_+ \leq |\delta_{j-1}| + |\delta_j|$. Using (1.4), summing, and $1 - e^{-d} \geq d/2$, gives

$$W_N^{\text{av}} \leq 6000 \frac{1 + e^{-d}}{1 - e^{-d}} \leq \frac{24000}{d} = 9.6 \cdot 10^{16} < 10^{17}.$$

For a fixed history, (6.3) bounds the incoming mean energy by $\sqrt{30} < 6$. The positive energy decrease in any branch cannot exceed that incoming energy. Summing and averaging proves (6.7). Signed heat and net work telescope, and $E_0 - E_* = q/(1-q) - q^2/(1-q^2) = q/(1-q^2)$. $\square$

The uniform constant in (6.6) concerns averaged-collision charging. It is not asserted for the stronger history-conditional convention.

7 Scope and reproducible evidence

The theorem concerns the specified initial state, target Gibbs law, and uniform time interval. It gives an explicit full-state rate; the constants are not intended to estimate practical cooling times. In particular, the proof does not establish a comparable rate for deterministic pulses or for arbitrary initial Fock-space states.

The closely related analysis in Ref. [1], Sec. IV.2 and Appendices B.2–B.3, uses a short-time expansion in Eqs. (66)–(70) and moment information in Eqs. (71)–(80). Our additional conclusion is the exact-channel trace-distance estimate (1.3), with a uniform infinite-sector argument. This comparison is specific to that source; comprehensive novelty assessment remains open.

The accompanying evidence package preserves the original statement, frozen proof M06-v1, arithmetic controls, and the separate-agent internal review. The standard-library checker verifies the closed-walk identities at $r = 1, \dots, 24$, exact rational safety margins, and rejection of an intentionally altered fourth moment. A second implementation uses 200-bit Arb enclosures for the transcendental constants [2]. These finite checks verify arithmetic interfaces. The all- r moment identities, infinite-chain inequalities, and initial-tail argument are the analytic proof, not consequences of finite enumeration. The package driver reproduces the checks in a clean copy and verifies input hashes first. Internal review within one model family is not external expert review.

AI-assisted research

This work was developed in an author-directed, AI-assisted research workflow. AI systems assisted with mathematical derivation, literature checks, computational verification, and manuscript drafting. The internal reviews and separate implementations described above were produced within this workflow and do not constitute external expert review.

References

- [1] W.-H. Png, X. Hu, and V. Scarani, *Sub-Bath Cooling in Bosonic Systems: Gaussian Constraints and Non-Gaussian Enhancements*, PRX Quantum **7**, 020316 (2026). doi:10.1103/3qnz-6bcz; arXiv:2512.10703v2.
- [2] F. Johansson, *Arb: Efficient Arbitrary-Precision Midpoint-Radius Interval Arithmetic*, IEEE Transactions on Computers **66**(8), 1281–1292 (2017). doi:10.1109/TC.2017.2690633; arXiv:1611.02831.