

Fixed-Photon-Number Codes Violate a Proposed Second-Order Pure-Loss Converse

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20 September 2026 (research preprint)

AI-assisted proof candidate, internally checked.

External expert review and publication priority remain unconfirmed.

Abstract

We give a counterexample to the occupation-constrained second-order upper bound recorded in QIQCOP problem `op_89fb664ba06ba5de`. For every fixed transmissivity $0 < \eta < 1$, energy parameter $N_S > 0$, and error tolerance $0 < \varepsilon < 1$, deterministic codes supported exactly on the $[nN_S]$ -photon sector achieve $ng(\eta N_S) + \sqrt{n\eta(1-\eta)N_S} \log_2(1 + 1/(\eta N_S)) \Phi^{-1}(\varepsilon) - \frac{3}{2} \log_2 n + O(1)$ bits. Their photon-cutoff leakage is zero. For $\varepsilon < 1/2$, the achieved second-order term exceeds the proposed converse by a positive multiple of $\sqrt{n}$. The proof combines established photon-sector isotropy and the binomial loss law with a finite-block Gram–Schmidt packing estimate using one codebook across all output sectors. We refute the catalog’s strengthened formulation, not a theorem attributed to its source papers. We do not prove an optimal replacement dispersion.

1 The occupation-constrained question

Let $\mathcal{N}_\eta$ be the single-mode bosonic pure-loss channel with vacuum environment, and let $\Pi_{\leq k}$ be the projection onto total photon number at most k in n modes. An M -message code consists of states ρ_m and an output POVM $(\Lambda_m)_{m=1}^M$. The catalog problem [1] requires

$$\frac{1}{M} \sum_m \text{Tr}[\Lambda_m \mathcal{N}_\eta^{\otimes n}(\rho_m)] \geq 1 - \varepsilon, \quad (1.1)$$

$$\frac{1}{M} \sum_m \text{Tr}[\Pi_{\leq [nN_S]} \rho_m] \geq 1 - \delta_n, \quad 0 \leq \delta_n \leq 2^{-cn}, \quad c > 0. \quad (1.2)$$

Write $M^*(n, \varepsilon, N_S, \eta, c)$ for the maximal code size with these properties. All logarithms in this paper are base two. Define

$$g(x) = (x+1) \log(x+1) - x \log x, \quad (1.3)$$

$$v(x) = x(x+1)[\log(1+1/x)]^2. \quad (1.4)$$

The proposed catalog upper bound is

$$\log M^* \leq ng(\eta N_S) + \sqrt{nv(\eta N_S)} \Phi^{-1}(\varepsilon) + O(\log n), \quad (1.5)$$

where Φ is the standard normal distribution function and parameters are fixed as $n \rightarrow \infty$.

Wilde, Renes, and Guha discuss a coherent-state construction under an occupation constraint [3, Sec. 6.1]; Wilde and Winter establish a first-order strong converse [4]. The catalog identifies (1.5) as its precise strengthened question. Our codewords need not be coherent or Gaussian, as permitted by (1.2).

Theorem 1. For every fixed $\eta \in (0, 1)$, $N_S > 0$, $\varepsilon \in (0, 1)$, and $c > 0$, there are deterministic codes satisfying (1.1)–(1.2) with $\delta_n = 0$ for every sufficiently large n , and

$$\begin{aligned} \log M_n &= ng(\eta N_S) + \sqrt{n\eta(1-\eta)N_S} \log\left(1 + \frac{1}{\eta N_S}\right) \Phi^{-1}(\varepsilon) \\ &\quad - \frac{3}{2} \log n + O(1). \end{aligned} \quad (1.6)$$

Consequently (1.5) is false for every fixed $\varepsilon < 1/2$.

2 Photon-sector inputs and a finite-block packing estimate

For a unit vector $u \in \mathbb{C}^n$ and an integer $j \geq 0$, set

$$|u; j\rangle = \frac{(\sum_{i=1}^n u_i a_i^\dagger)^j}{\sqrt{j!}} |0\rangle. \quad (2.1)$$

This normalized state lies in the j -photon space $\mathcal{H}_j$, whose dimension is $d_j = \binom{n+j-1}{j}$. Put $k = \lceil nN_S \rceil$. The beamsplitter transformation with vacuum environment gives

$$\mathcal{N}_\eta^{\otimes n}(|u; k\rangle\langle u; k|) = \sum_{l=0}^k p_l |u; l\rangle\langle u; l|, \quad p_l = \binom{k}{l} \eta^l (1-\eta)^{k-l}. \quad (2.2)$$

Indeed, the joint output has amplitudes $\sqrt{p_l}$ multiplying $|u; l\rangle_B |u; k-l\rangle_E$; distinct environment photon sectors are orthogonal. Every codeword has exactly k photons and hence zero leakage.

The isotropy and binomial loss ingredients are established: see Fanizza et al. [2, Sec. 3.2, Eq. (9), and Sec. 5, Eq. (42)]. For completeness, if u is Haar-uniform on the complex unit sphere, then

$$\mathbb{E}_u |u; l\rangle\langle u; l| = \frac{I_{\mathcal{H}_l}}{d_l}. \quad (2.3)$$

To verify this directly, index the occupation basis by $r \in \mathbb{N}_0^n$ with $\sum_i r_i = l$. Coordinate phase invariance eliminates off-diagonal averages. The diagonal amplitude squared is $(l! / \prod_i r_i!) \prod_i |u_i|^{2r_i}$, and

$$\mathbb{E} \prod_i |u_i|^{2r_i} = \frac{(n-1)! \prod_i r_i!}{(n+l-1)!}. \quad (2.4)$$

The latter identity follows by writing a vector of independent standard complex Gaussians as $z = Ru$: direction and radius are independent, $\mathbb{E} \prod_i |z_i|^{2r_i} = \prod_i r_i!$, and $\mathbb{E} R^{2l} = (n+l-1)!/(n-1)!$.

Lemma 2. For any integers $n \geq 2$, $k \geq 0$, $M \geq 1$, and $0 \leq t \leq k$, there is a code of M inputs of the form $|u; k\rangle$ and a full output POVM with average error at most

$$\mathbb{P}(L < t) + \frac{M-1}{2d_t}, \quad L \sim \text{Bin}(k, \eta). \quad (2.5)$$

Proof. Draw $u_1, \dots, u_M$ independently and uniformly on the unit sphere. For each $l \geq t$, write $\psi_m = |u_m; l\rangle$ and let $P_{m-1,l}$ project onto the span of the preceding $\psi_1, \dots, \psi_{m-1}$. Set $w_m = (I - P_{m-1,l})\psi_m$. If $w_m \neq 0$, let $Q_{m,l}$ project onto $w_m/\|w_m\|$; otherwise set $Q_{m,l} = 0$. These projections are mutually orthogonal and in either case

$$\langle \psi_m, Q_{m,l} \psi_m \rangle = 1 - \langle \psi_m, P_{m-1,l} \psi_m \rangle. \quad (2.6)$$

By independence and (2.3), the conditional expected error is $\text{rank}(P_{m-1,l})/d_l$, which is at most $(m-1)/d_l$. Thus the expected average error of this incomplete sector measurement is at most $(M-1)/(2d_l)$. No uniform distribution on the full sphere of $\mathcal{H}_l$ is assumed.

Use the same codebook in every sector and set $Q_m = \bigoplus_{l=t}^k Q_{m,l}$, with zero action elsewhere. Since $\sum_m Q_m \leq I$ on the entire output Fock space, the operators

$$\Lambda_1 = Q_1 + I - \sum_m Q_m, \quad \Lambda_m = Q_m \quad (m \geq 2) \quad (2.7)$$

form a POVM. Completion cannot reduce success. Linearity of expectation, (2.2), and the monotonicity of d_l in l show that its expected average error is at most $\mathbb{P}(L < t) + \sum_{l=t}^k p_l(M - 1)/(2d_l)$, bounded by (2.5). At least one deterministic codebook achieves that bound. $\square$

3 Quantile choice and the second-order rate

For sufficiently large n let $q_n = \varepsilon - 1/n$ and define t_n as the smallest integer with $\mathbb{P}(L \leq t_n) \geq q_n$, where $L \sim \text{Bin}(\lceil nN_S \rceil, \eta)$. Set

$$M_n = \left\lfloor \frac{d_{t_n}}{n} \right\rfloor. \quad (3.1)$$

The dimension is exponentially large, as proved below. For sufficiently large n , $M_n \geq 1$, and the bound in Lemma 2 is strictly less than $q_n + 1/(2n) = \varepsilon - 1/(2n)$. This supplies the required code.

Let $\mu_n = k\eta$, $\sigma_n = \sqrt{k\eta(1-\eta)}$, and $z = \Phi^{-1}(\varepsilon)$. The Berry–Esseen theorem for Bernoulli sums bounds the error in the normal distribution approximation uniformly by $C_\eta/\sqrt{k}$. The quantile is interior for sufficiently large n . Applying the approximation at t_n and $t_n - 1$ and using minimality gives

$$\Phi^{-1}\left(q_n - \frac{C_\eta}{\sqrt{k}}\right) \leq \frac{t_n - \mu_n}{\sigma_n} < \Phi^{-1}\left(q_n + \frac{C_\eta}{\sqrt{k}}\right) + \frac{1}{\sigma_n}. \quad (3.2)$$

The derivative of Φ^{-1} is bounded on a neighborhood of the fixed ε . Since $k = nN_S + O(1)$, it follows that

$$t_n = n\eta N_S + \sqrt{n\eta(1-\eta)N_S} z + O(1). \quad (3.3)$$

Write $x = \eta N_S > 0$ and $a = \sqrt{\eta(1-\eta)N_S} z$. Uniform Stirling estimates for t_n/n near x yield

$$\log d_{t_n} = ng(t_n/n) - \frac{1}{2} \log n + O(1) \quad (3.4)$$

$$= ng(x) + a\sqrt{n} \log(1 + 1/x) - \frac{1}{2} \log n + O(1). \quad (3.5)$$

Here $g'(x) = \log(1 + 1/x)$, and the quadratic Taylor remainder contributes only $O(1)$. In particular d_{t_n} grows exponentially. Taking the integer part in (3.1) changes its logarithm by $o(1)$ relative to $\log d_{t_n} - \log n$, proving (1.6).

4 Contradiction and interpretation

The positive second-order coefficients in the achieved and proposed rates are, respectively,

$$A = \sqrt{\eta(1-\eta)N_S} \log(1 + 1/x), \quad B = \sqrt{x(1+x)} \log(1 + 1/x). \quad (4.1)$$

Since $1 - \eta < 1 + \eta N_S$, we have $A < B$. For $\varepsilon < 1/2$, $z < 0$, so $Az > Bz$. If (1.5) held, comparison with feasible codes would imply $(B - A)|z|\sqrt{n} \leq O(\log n)$, a contradiction. For example, $\eta = 1/2$, $N_S = 2$, and $\varepsilon = \Phi(-1)$ give

$$\log M_n = 2n - \sqrt{n/2} - \frac{3}{2} \log n + O(1), \quad (4.2)$$

whereas the proposed upper bound is $2n - \sqrt{2n} + O(\log n)$. The contradiction holds for every prescribed $c > 0$ because leakage is zero.

The result is an existence theorem with a codebook-dependent decoder. No efficient construction, practical decoding scheme, or matching second-order converse is asserted. The familiar isotropy and loss laws are credited above; the manuscript isolates their operational packing consequence for this specific occupation-constrained statement. Finite checks of the shell moments, Kraus output, and decoder algebra support implementation accuracy but do not establish the asymptotic theorem.

Verification, provenance, and scope

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