

A three-qubit counterexample to the ordinary Petz recovery bound

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The Fawzi–Renner inequality relates conditional mutual information to the fidelity of recovery after a subsystem is discarded. The ordinary Petz map gives exact recovery when the conditional mutual information vanishes, but it need not satisfy the same quantitative bound away from equality. We give an explicit rank-two state of three qubits for which the conditional mutual information is 0.06709806646 bits, whereas the ordinary-Petz recovery term is 0.07137976270 bits. All coefficients are specified as exact decimals, and the negative gap is certified by rational arithmetic with explicit bounds on the matrix square roots and logarithms. The example also yields full-rank counterexamples by continuity. An average of rotated Petz maps satisfies the known recovery inequality on the same state. The problem was solved by Claude Opus 5, and the solution was independently verified by GPT 6 Astra.

I. INTRODUCTION

Conditional mutual information connects the correlations in a tripartite state to the possibility of recovering a discarded subsystem. If $I(A; B | C)_\rho = 0$, the correlations involving A can be reconstructed from ρ_{BC} by acting on C alone. The Petz map realizes this exact reconstruction [1, 2]. Fawzi and Renner showed that a quantitative statement survives when the conditional mutual information is positive: there exists a recovery channel whose fidelity is controlled by $I(A; B | C)_\rho$ [3].

The choice of recovery channel matters. Bounds obtained by optimizing over a rotation parameter [4], and by averaging suitable rotated Petz maps [5, 6], do not by themselves establish the same bound for the unrotated map. The latter is a particularly natural candidate: it is explicit, depends only on the appropriate marginal, and already solves the equality case. The ordinary-Petz question asks whether these properties are enough for the fidelity bound as well [7].

We give a negative answer with an explicit state on $\mathbb{C}_A^2 \otimes \mathbb{C}_B^2 \otimes \mathbb{C}_C^2$. A rational-arithmetic certificate establishes the sign of the violation using matrices of order at most eight.

Throughout, logarithms are base two unless written as $\ln$. For a density operator τ , let $S(\tau) = -\text{Tr}(\tau \log_2 \tau)$, and write

$$I(A; B | C)_\rho = S(\rho_{AC}) + S(\rho_{BC}) - S(\rho_C) - S(\rho_{ABC}). \quad (1)$$

We use the squared fidelity $F(\tau, \omega) = \|\sqrt{\tau}\sqrt{\omega}\|_1^2$. The ordinary Petz map associated with ρ_{AC} and the partial trace over A is

$$\mathcal{P}_{C \rightarrow AC}^\rho(X) = \rho_{AC}^{1/2} [I_A \otimes \rho_C^{-1/2} X \rho_C^{-1/2}] \rho_{AC}^{1/2}, \quad (2)$$

where inverses are taken on $\text{supp } \rho_C$. Set $\rho_{\text{rec}} = (\text{id}_B \otimes \mathcal{P}_{C \rightarrow AC}^\rho)(\rho_{BC})$, with its tensor factors reordered as ABC . The proposed inequality is

$$\Delta(\rho) := I(A; B | C)_\rho + \log_2 F(\rho_{ABC}, \rho_{\text{rec}}) \geq 0. \quad (3)$$

This is a specialization of a fidelity remainder for relative-entropy data processing. Indeed, for $\sigma = \rho_{AC} \otimes I_B/d_B$ and $\mathcal{N} = \text{Tr}_A$, with the identity on the spectator system understood,

$$D(\rho \| \sigma) - D(\mathcal{N}(\rho) \| \mathcal{N}(\sigma)) = I(A; B | C)_\rho, \quad (4)$$

and the associated Petz map is precisely the one in Eq. (2), extended to B . Bhattacharya disproved the general remainder bound with a qubit pinching channel [8]. That example allows an independently chosen reference state. Here the reference is constrained by a marginal of the state being recovered, so a counterexample must satisfy this additional relation.

II. AN EXPLICIT STATE

The construction is specified by two positive weights and two vectors. In the computational basis $|abc\rangle$, take

$$w_1 = 0.0383477362, \quad w_2 = 0.9616522638, \quad (5)$$

and let v_1, v_2 have the coefficients in Table I. Every displayed decimal in that table and in Eq. (5) is part of the exact definition. In particular, the vectors are not assumed to remain exactly orthonormal after decimal truncation. Define

$$M = w_1 v_1 v_1^\dagger + w_2 v_2 v_2^\dagger, \quad Z = \text{Tr } M, \quad \rho = \frac{M}{Z}. \quad (6)$$

The normalization is the exact rational number

$$Z = 0.9999999997682180845905542546. \quad (7)$$

Since the weights are positive and the vectors linearly independent, Eq. (6) defines a density operator of rank two.

TABLE I. Exact coefficients of the two vectors used in Eq. (6). The row labels fix the tensor order as ABC .

abc	$\langle abc v_1\rangle$	$\langle abc v_2\rangle$
000	-0.6104357345	-0.0465530650
001	$0.0063927953 - 0.0044244357i$	$-0.0377544806 - 0.0978816076i$
010	$0.0026075606 - 0.0150331220i$	$0.8582239838 - 0.4826969158i$
011	$-0.0001180574 - 0.0002823919i$	$-0.0132291583 + 0.0104067043i$
100	$-0.0118114830 + 0.0054943442i$	$0.0018681142 + 0.0014701336i$
101	$-0.0003861017 - 0.0098540922i$	$-0.0012199090 + 0.0109247234i$
110	$0.7842448379 - 0.1079189498i$	$-0.0488402257 - 0.0085549482i$
111	$-0.0061499192 + 0.0091457466i$	$-0.0582671635 - 0.1049688866i$

Proposition 1. *The state in Eq. (6) violates Eq. (3). Its gap obeys the certified bounds*

$$-0.00428169624320 \leq \Delta(\rho) \leq -0.00428169624091 \quad (8)$$

in bits. The marginals ρ_{AC} , ρ_{BC} , and ρ_C are positive definite.

Table II gives the numerical values entering the inequality. The right-hand side exceeds the conditional mutual information by about 6.38%. The relevant marginal spectra are

$$\begin{aligned} \text{spec}(\rho_C) &\simeq (0.02457815, 0.97542185), \\ \text{spec}(\rho_{AC}) &\simeq (0.00026275, 0.01062034, 0.02407114, 0.96504577), \\ \text{spec}(\rho_{BC}) &\simeq (0.00020683, 0.01395404, 0.01427343, 0.97156569). \end{aligned} \quad (9)$$

Thus the inverse in Eq. (2) is an ordinary inverse. The recovered state is also positive definite; its smallest eigenvalue is approximately 2.24×10^{-6} .

TABLE II. Evaluation on the normalized state of Eq. (6). Entropies and the recovery term are in bits; F is dimensionless. Values shown here are rounded from an 80-digit computation.

Quantity	Value
$S(AC)$	0.2517192097613853
$S(BC)$	0.2164694791674901
$S(C)$	0.1664259748231664
$S(ABC)$	0.2346646476485522
$I(A; B C)$	0.0670980664571567
$F(\rho, \rho_{\text{rec}})$	0.9517273509565518
$-\log_2 F(\rho, \rho_{\text{rec}})$	0.0713797626992107

III. VERIFICATION AND AN EXACT-ARITHMETIC CERTIFICATE

A direct matrix implementation and a separate Choi-matrix implementation give recovered states agreeing entrywise to 3.4×10^{-16} . Calculations at 50 and 80 decimal digits agree in more than 40 displayed digits. These checks test the implementation; the certificate below accounts for numerical error and establishes the strict inequality.

The rank-two factorization makes the fidelity calculation particularly simple. Let

$$G = Z^{-1/2} \begin{pmatrix} \sqrt{w_1} v_1 & \sqrt{w_2} v_2 \end{pmatrix}, \quad K = G^\dagger \rho_{\text{rec}} G. \quad (10)$$

The nonzero eigenvalues of $\sqrt{\rho_{\text{rec}}} \rho \sqrt{\rho_{\text{rec}}}$ are those of the 2×2 positive matrix K . Consequently,

$$F(\rho, \rho_{\text{rec}}) = (\text{Tr } \sqrt{K})^2 = \text{Tr } K + 2\sqrt{\det K}. \quad (11)$$

Similarly, the entropy of ρ can be computed from $G^\dagger G$. This avoids taking square roots of the six numerical zero eigenvalues of an 8×8 rank-two matrix.

For the certificate, all entries of ρ and its marginals are first formed as exact complex rationals. Each eigenvalue needed for an entropy is enclosed in an interval of width 3×10^{-14} . The characteristic polynomial has opposite signs at the two rational endpoints. The intervals are disjoint, and their number equals the polynomial degree, so they exhaust its roots. For $S(ABC)$, only a quadratic is needed: its roots have sum one and product $w_1 w_2 \det(V^\dagger V)/Z^2$, where $V = (v_1 \ v_2)$.

Next, 40-decimal Hermitian rational approximants A_0 and T_0 are formed for $\rho_{AC}^{1/2}$ and $\rho_C^{-1/2}$. Exact determinant checks give

$$\rho_{AC} > 10^{-4} I, \quad \rho_C > \frac{1}{50} I, \quad A_0 > \frac{1}{100} I, \quad T_0 > 0, \quad (12)$$

and exact rational sums of squares establish

$$\|A_0^2 - \rho_{AC}\|_F < 10^{-30}, \quad \|T_0^2 - \rho_C^{-1}\|_F < 10^{-30}. \quad (13)$$

For positive definite X, Y , the identity $X(X - Y) + (X - Y)Y = X^2 - Y^2$ and its Sylvester integral solution imply

$$\|X - Y\| \leq \frac{\|X^2 - Y^2\|}{\lambda_{\min}(X) + \lambda_{\min}(Y)}. \quad (14)$$

Here $\|\cdot\|$ is the operator norm. Equations (12)–(14) bound the two approximation errors by 5×10^{-29} and 10^{-30} , respectively. Propagating them through Eq. (2) bounds the operator-norm error in the recovered state by 10^{-24} .

The remaining scalar operations are enclosed with rational arithmetic. Square roots are bounded by integer-square-root comparisons. For logarithms, binary scaling reduces the argument to $x \in [1, 2]$; with $z = (x - 1)/(x + 1)$, use

$$0 \leq \ln x - 2 \sum_{j=0}^{N-1} \frac{z^{2j+1}}{2j+1} \leq \frac{2z^{2N+1}}{(2N+1)(1-z^2)}, \quad N = 30. \quad (15)$$

The same formula encloses $\ln 2$. Interval evaluation then gives

$$\begin{aligned} I(A; B | C)_\rho &\in [0.06709806645602, 0.06709806645830], \\ -\log_2 F(\rho, \rho_{\text{rec}}) &\in [0.07137976269921, 0.07137976269922]. \end{aligned} \quad (16)$$

Evaluating the difference before outward decimal rounding yields Eq. (8), proving Proposition 1. The certificate uses numerical routines only to propose intervals and approximants; the inequalities accepting them are checked exactly.

As a separate check on the channel, its Kraus operators satisfy $\|\sum_j L_j^\dagger L_j - I_C\|_F < 10^{-15}$ in double precision, and $\|\mathcal{P}_{C \rightarrow AC}^\rho(\rho_C) - \rho_{AC}\|_F < 5 \times 10^{-16}$. These tests check trace preservation and recovery of the reference marginal, rather than inferring either property from the fidelity alone.

IV. ROTATED RECOVERY AND FULL-RANK STATES

The counterexample is consistent with the known recovery theorem. To compare the maps with a fixed convention, define

$$\mathcal{P}_t(X) = \rho_{AC}^{(1+it)/2} [I_A \otimes \rho_C^{-(1+it)/2} X \rho_C^{-(1-it)/2}] \rho_{AC}^{(1-it)/2}, \quad (17)$$

so that $\mathcal{P}_0 = \mathcal{P}_{C \rightarrow AC}^\rho$. The universal average can be written as [6]

$$\bar{\mathcal{P}} = \int_{\mathbb{R}} \beta_0(t) \mathcal{P}_t dt, \quad \beta_0(t) = \frac{\pi/2}{\cosh(\pi t) + 1}. \quad (18)$$

An overall reversal of the rotation parameter makes no difference to this average. Equation (17) fixes the parameter convention used below.

For the state above, numerical integration gives

$$-\log_2 F(\rho, (\text{id}_B \otimes \bar{\mathcal{P}})(\rho_{BC})) \simeq 0.05042295252 < I(A; B | C)_\rho. \quad (19)$$

Gauss–Legendre integration with 240 and 480 nodes on $[-12, 12]$ agrees to approximately 10^{-14} ; the omitted probability mass is below 8.5×10^{-17} . Individual rotations can be better or worse than $t = 0$. For example, the recovery term is approximately 0.02770497162 at $t = -0.8$, but 0.07355761393 at $t = -1.6$. Thus zero rotation need not be the worst member of the family. What fails is the claim that this particular member always satisfies Eq. (3).

The violation is also not confined to the boundary of the state space.

Corollary 1. *There exist positive definite three-qubit states that violate Eq. (3).*

Proof. At the state in Eq. (6), ρ_C is positive definite. The matrix functions in Eq. (2), the fidelity, and the entropies are therefore continuous in a neighborhood of this state. Since $\Delta(\rho) < 0$ strictly, the same holds for

$$\rho_\epsilon = (1 - \epsilon)\rho + \epsilon I_8/8 \quad (20)$$

for all sufficiently small $\epsilon > 0$. Each such state is positive definite. $\square$

As a numerical illustration, $\epsilon = 10^{-4}$ gives $I(A; B | C) \simeq 0.06664048223$ and a recovery term of 0.06844288740 bits, leaving a gap of approximately -0.00180240517 bits. Unlike the main certificate, these last values are reported only as floating-point evaluations.

V. DISCUSSION

The ordinary Petz map retains its exact recovery property at vanishing conditional mutual information, but that property does not extend to the proposed fidelity bound. Three qubits already suffice to separate the two statements. The strict negative gap and the full-rank perturbations also show why an explanation based on support conventions cannot account for the example.

The scope of the conclusion is specific. The Fawzi–Renner recovery theorem remains valid, and the averaged rotated map satisfies it on this state. The example does not establish that averaging is the only way to obtain an appropriate recovery channel, nor that any prescribed nonzero rotation fails universally. It also gives no estimate of the largest possible violation in this or higher dimensions. A simpler family with a transparent analytic structure would help explain which correlations make the ordinary map fall short. For the present question, the explicit state and its certified negative gap are sufficient.

Reproducibility. The accompanying source archive includes a standalone numerical verification, the rational-arithmetic certificate, and their output files. The certificate requires Python with NumPy, mpmath, and SymPy. All computations start from the coefficients printed in Table I.

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- [1] D. Petz, Sufficient subalgebras and the relative entropy of states of a von Neumann algebra, *Commun. Math. Phys.* **105**, 123–131 (1986).
 - [2] D. Petz, Sufficiency of channels over von Neumann algebras, *Q. J. Math.* **39**, 97–108 (1988).
 - [3] O. Fawzi and R. Renner, Quantum conditional mutual information and approximate Markov chains, *Commun. Math. Phys.* **340**, 575–611 (2015), [arXiv:1410.0664](https://arxiv.org/abs/1410.0664).
 - [4] M. M. Wilde, Recoverability in quantum information theory, *Proc. R. Soc. A* **471**, 20150338 (2015), [arXiv:1505.04661](https://arxiv.org/abs/1505.04661).
 - [5] D. Sutter, M. Tomamichel, and A. W. Harrow, Strengthened monotonicity of relative entropy via pinched Petz recovery map, *IEEE Trans. Inf. Theory* **62**, 2907–2913 (2016), [arXiv:1507.00303](https://arxiv.org/abs/1507.00303).
 - [6] M. Junge, R. Renner, D. Sutter, M. M. Wilde, and A. Winter, Universal recovery maps and approximate sufficiency of quantum relative entropy, *Ann. Henri Poincaré* **19**, 2955–2978 (2018), [arXiv:1509.07127](https://arxiv.org/abs/1509.07127).
 - [7] Quantum Information and Quantum Computation Open Problem Zoo, Ordinary-Petz recovery bound for conditional mutual information, problem `op_87c77263c8bab523`, https://qiqc-op.com/problem/op_87c77263c8bab523/.
 - [8] S. Bhattacharya, Approximate recoverability and the quantum data processing inequality, [arXiv:2309.02074v3](https://arxiv.org/abs/2309.02074v3) (2025).