The spectral gap of random Pauli rotations on four qubits is 1/15

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28 September 2026

Abstract

Each step of the random-Pauli-rotation walk on $\mathrm{SU}(2^n)$ applies $e^{i\theta P}$, with P a uniformly random non-identity n-qubit Pauli operator and θ uniform on $[0, 2\pi)$. Baer and Haah proved $(4^n + 16)/(16(4^n - 1)) \leq \Delta_n \leq 2^n(2^n - 3)/(8(4^n - 1))$ for its spectral gap ($n \geq 3$) and conjectured that the upper bound, which comes from the fourth moment, is exact. We show that the lower bound is attained at $n = 4$: $\Delta_4 = 1/15$, not $26/255$. A slowest mode lies in the unbalanced irreducible representation $\Lambda^8\mathbb{C}^{16}$. The sum w of the 30 wedge vectors indexed by the affine hyperplanes of $\mathbb{F}_2^4$ is Clifford invariant, and for each Z-type Pauli operator 28 of its 30 terms carry zero charge, so w is an eigenvector with eigenvalue $14/15$; equivalently, $f(U) = \sum_A \det U[A, B_0]$ is an explicit zero-mean eigenfunction of the walk with eigenvalue $14/15$. The matching upper bound follows from a Casimir identity and from a selection rule, which also shows that no irreducible representation λ with $8 \nmid |\lambda|$ violates the conjectured value. A standard-library Python program checks every finite step of the proof in exact arithmetic.

1 Introduction

Let $d = 2^n$ and $\mathcal{P}_n = \{I, X, Y, Z\}^{\otimes n} \setminus \{I^{\otimes n}\}$. One step of the random-Pauli-rotation walk chooses $P \in \mathcal{P}_n$ and $\theta \in [0, 2\pi)$ uniformly and independently and multiplies by $e^{i\theta P}$, which lies in $\mathrm{SU}(d)$ because $\operatorname{Tr} P = 0$. On $L^2(\mathrm{SU}(d))$ with Haar measure, the Markov operator and its spectral gap are [7]

$$(K_n f)(U) = \mathbb{E}_{P,\theta}\, f(e^{i\theta P}U), \qquad \Delta_n = 1 - \big\|K_n|_{L^2_0}\big\|, \tag{1}$$

where L^2_0 consists of the functions with zero Haar mean. Haah, Liu and Tan used random Pauli rotations to construct efficient approximate unitary designs, with a gap $\Omega(1/t)$ for the t-th moment operator [2]. Baer and Haah proved that Δ_n is bounded below by a constant and conjectured [1, Conj. 3.48] that for $n \geq 3$

$$\Delta_n = \frac{2^n(2^n - 3)}{8(4^n - 1)}, \tag{2}$$

with the irreducible representation of highest weight $(1^4, 0^{d-8}, -1^4)$ tight; their numerical evidence used the representation $\tau_{4,4}$ for $n \leq 3$ [1, Remark 3.47]. What they proved is

$$\frac{4^n + 16}{16(4^n - 1)} \leq \Delta_n \leq \frac{2^n(2^n - 3)}{8(4^n - 1)} \qquad (n \geq 3). \tag{3}$$

The lower bound holds for every n and is tight for $n \leq 3$ [1, Thm. 3.16, Prop. 3.17]; the upper bound comes from the fourth-moment representation $U^{\otimes 4} \otimes \bar{U}^{\otimes 4}$ [1, Prop. 3.17]. Thus $\Delta_3 = 5/63$ and $1/15 \leq \Delta_4 \leq 26/255$. The QIQC Open Problem Zoo asks whether Eq. (2) holds for every $n \geq 4$ [7].

The gap of [1, Def. 1.1] is an infimum over all finite-dimensional unitary representations of $\mathrm{SU}(d)$, and by the Peter–Weyl theorem it equals Δ_n (Lemma 2). Both therefore include the *unbalanced* irreducible representations, on which the centre of $\mathrm{SU}(d)$ acts nontrivially and which occur in no $U^{\otimes t} \otimes \bar{U}^{\otimes t}$. At $n = 4$ one of them mixes more slowly than the fourth moment.

Theorem 1. *Let $n = 4$, let $M = \mathbb{E}_{P,\theta}\Lambda^8(e^{i\theta P})$ act on $\Lambda^8\mathbb{C}^{16}$, and let $w = \sum_A e_A$ be the sum, over the 30 affine hyperplanes A of $\mathbb{F}_2^4$, of the wedge products e_A of their basis vectors in ascending binary order (Section 3). Then $Mw = \frac{14}{15}w$ and $\|M\| = \frac{14}{15}$. Consequently $\Delta_4 \le 1/15 = 17/255 < 26/255 = 2^4(2^4-3)/(8(4^4-1))$, so Eq. (2) fails at $n = 4$, and Conjecture 3.48 of [1] is false. Together with [1, Thm. 3.16], $\Delta_4 = 1/15$.*

The answer to the recorded question is therefore no, and the Baer–Haah lower bound is tight at $n = 4$. The representation $\Lambda^8\mathbb{C}^{16}$ is a bottleneck; we do not claim that it is the only one. The eigenvector yields the explicit eigenfunction $f(U) = \sum_A \det U[A, B_0]$, and $K_4 f = \frac{14}{15}f$ can be checked at any U with finitely many determinants (Corollary 9).

The lower bound on $\|M\|$ combines a count for Z-type Pauli operators (Lemma 3) with the Clifford invariance of w (Proposition 6), whose key input is the self-duality of the Reed–Muller code $\mathrm{RM}(1,3)$. The upper bound follows from a quadratic Casimir identity (Proposition 8) and, independently, from a selection rule in terms of the 2-adic valuation of $|\lambda|$ (Lemma 10).

2 Setup and reduction

Conventions. We label the computational basis of $\mathbb{C}^d = (\mathbb{C}^2)^{\otimes n}$ by $x \in \mathbb{F}_2^n$, qubit j carrying the bit x_j, and order $\mathbb{F}_2^n$ by the integer $\sum_j x_j 2^j$; this is the *binary order.* The operator $Z^s = \prod_j Z_j^{s_j}$ acts by $Z^s e_x = (-1)^{s\cdot x} e_x$, and the support vector $s(P) \in \mathbb{F}_2^n$ of $P \in \mathcal{P}_n$ has $s(P)_j = 1$ exactly when the factor of P on qubit j is not I. We use the gates H_j, $S_j = \mathrm{diag}(1, i)_j$ and $\mathrm{CNOT}_{jk}\colon e_x \mapsto e_{x + x_j\varepsilon_k}$, where ε_k is the k-th unit vector, and the relations $HXH = Z$ and $S^\dagger Y S = X$.

Representations. An irreducible representation ρ of $\mathrm{SU}(d)$ has a highest weight $\lambda \in \mathbb{Z}^d$, $\lambda_1 \ge \cdots \ge \lambda_d$, defined up to adding a constant vector; put $|\lambda| = \sum_i \lambda_i \bmod d$. Taking $\lambda_d \ge 0$, ρ extends to a polynomial representation of $\mathrm{GL}(d)$ of some degree $k \equiv |\lambda| \pmod d$, whose weights $\mu \in \mathbb{Z}_{\ge 0}^{\mathbb{F}_2^n}$ all satisfy $\sum_x \mu_x = k$ [5]. A scalar $\omega \cdot 1 \in \mathrm{SU}(d)$ acts as $\omega^{|\lambda|}$, so the centre acts trivially exactly on the *balanced* representations, $|\lambda| = 0$, which are those occurring in some $U^{\otimes t} \otimes \bar{U}^{\otimes t}$. For a Hermitian traceless P let $H_P = d\rho(P)$, so that $\rho(e^{i\theta P}) = e^{i\theta H_P}$ and $\rho(C)H_P\rho(C)^{-1} = H_{CPC^{-1}}$. The eigenvalues of H_P are the *P-charges*; Π_0^P is the orthogonal projector onto $\ker H_P$, and $\Pi_0^{-P} = \Pi_0^P$.

Exterior powers. For $1 \le k < d$, $\Lambda^k\mathbb{C}^d$ is the irreducible representation of highest weight $(1^k, 0^{d-k})$ [5]. For a k-subset $B = \{b_1 < \cdots < b_k\}$ of $\mathbb{F}_2^n$ let $e_B = e_{b_1} \wedge \cdots \wedge e_{b_k}$. These vectors form an orthonormal basis, and $\langle e_A, \Lambda^k(U)e_B\rangle = \det U[A, B]$ by the Cauchy–Binet formula, the minor with rows A and columns B in ascending order. On antisymmetric tensors $H_P = \sum_{i=1}^k P^{(i)}$, where $P^{(i)}$ acts on the i-th factor, and a diagonal matrix D acts by $H_D e_B = (\sum_{x\in B} D_{xx})\, e_B$. So every e_B is a weight vector, and

$$H_{Z^s} e_B = c_s(B)\, e_B, \qquad c_s(B) = \sum_{x \in B} (-1)^{s\cdot x}. \tag{4}$$

For $k = 8$ and $d = 16$, each $P \in \mathcal{P}_4$ has the eigenvalues ± 1 with multiplicity 8, and the P-charges are sums of eight of them over distinct eigenvectors, so they lie in $\{-8, -6, \ldots, 8\}$.

Lemma 2 (Reduction)**.** *Let ρ be an irreducible unitary representation of $\mathrm{SU}(d)$ on V and $M_\rho = \mathbb{E}_{P,\theta}\,\rho(e^{i\theta P})$. (a) The P-charges are integers, so $M_\rho = \mathbb{E}_{P\in\mathcal{P}_n}\Pi_0^P$ and $0 \le M_\rho \le 1$. (b) For $u, v \in V$ the matrix coefficient $f(U) = \langle u, \rho(U)v\rangle$ satisfies $(K_n f)(U) = \langle M_\rho u, \rho(U)v\rangle$. (c) $\|K_n|_{L_0^2}\| = \sup_\rho \|M_\rho\|$ over the nontrivial irreducible ρ, and Δ_n equals the gap $\Delta(\nu_{\mathrm{RPR}}, \mathrm{SU}(2^n))$ of [1, Def. 1.1]. In particular $\Delta_n \le 1 - \|M_\rho\|$ for every nontrivial ρ.*

Proof. (a) Since $P^2 = 1$, $e^{2\pi i H_P} = \rho(e^{2\pi i P}) = 1$, so the charges are integers and the θ-average of $e^{i\theta H_P}$ is Π_0^P. (b) M_ρ is Hermitian by (a), and $(K_n f)(U) = \mathbb{E}\,\langle u, \rho(e^{i\theta P})\rho(U)v\rangle = \langle M_\rho^\dagger u, \rho(U)v\rangle$. (c) By the Peter–Weyl theorem [4], $L^2(\mathrm{SU}(d))$ is the orthogonal sum, over the irreducible ρ, of the spaces of matrix coefficients of ρ, each isomorphic to $V_\rho \otimes V_\rho^*$, and the trivial ρ gives the constants. By (b), K_n acts on the ρ-block as $M_\rho \otimes 1$. In [1, Def. 1.1] the gap is $\inf_\rho(1 - \|M(\nu,\rho) - M(\mu,\rho)\|)$ over all finite-dimensional unitary representations, where $M(\nu,\rho) = \mathbb{E}_{g\sim\nu}\,\rho(g)$ and μ is the Haar measure. On trivial components both operators are the identity; on a nontrivial irreducible component ρ_i, $M(\mu,\rho) = 0$ and $M(\nu,\rho) = M_{\rho_i}$. So the infimum equals $1 - \sup\|M_\rho\|$ over the nontrivial irreducible ρ. □

3 The witness

From now on $n = 4$, $\rho = \Lambda^8$ and $M = \mathbb{E}_{P\in\mathcal{P}_4}\Pi_0^P$ (Lemma 2(a)). For $t \in \mathbb{F}_2^4 \setminus \{0\}$ and $c \in \mathbb{F}_2$, $A_{t,c} = \{x \in \mathbb{F}_2^4 : t\cdot x = c\}$ is the affine hyperplane with normal t. These are 30 distinct sets of 8 points, and

$$w = \sum_{t\neq 0}\sum_{c\in\mathbb{F}_2} e_{A_{t,c}}, \qquad \|w\|^2 = 30, \tag{5}$$

where every e_A is formed in ascending binary order, as in Section 2.

Lemma 3 (Z-type charges)**.** *Let $s \in \mathbb{F}_2^4 \setminus \{0\}$. Then $c_s(A_{t,c}) = (-1)^c\, 8$ if $t = s$, and $c_s(A_{t,c}) = 0$ if $t \neq s$. Hence $\Pi_0^{Z^s} w = w - e_{A_{s,0}} - e_{A_{s,1}}$ and $\langle w, \Pi_0^{Z^s} w\rangle = 28$.*

Proof. If $t = s$, then $s\cdot x = c$ on $A_{t,c}$. If $t \neq s$, then s and t are linearly independent, so $x \mapsto (t\cdot x, s\cdot x)$ maps $\mathbb{F}_2^4$ onto $\mathbb{F}_2^2$ with fibres of size 4, and $s\cdot x$ takes each value on exactly 4 points of $A_{t,c}$. By (4) each e_A is an eigenvector of H_{Z^s} with eigenvalue $c_s(A)$, so $\Pi_0^{Z^s}$ keeps exactly the 28 terms of w with $c_s(A) = 0$. □

The signs of the wedges e_A are controlled by the following lemma.

Lemma 4 (Order lemma)**.** *Let A be an affine hyperplane of $\mathbb{F}_2^4$ and $\varphi\colon \mathbb{F}_2^3 \to A$ an affine bijection. Then $e_{\varphi(0)} \wedge e_{\varphi(1)} \wedge \cdots \wedge e_{\varphi(7)} = e_A$, where $0, 1, \dots, 7$ lists $\mathbb{F}_2^3$ in binary order. Consequently, if g is an affine bijection of $\mathbb{F}_2^4$ and $\Pi_g e_x = e_{g(x)}$, then $\Lambda^8(\Pi_g)e_A = e_{g(A)}$ for every affine hyperplane A, and $\Lambda^8(\Pi_g)w = w$.*

Proof. Let $A = A_{t,c}$, let p be the smallest index with $t_p = 1$, and let $\psi\colon A \to \mathbb{F}_2^3$ delete x_p and keep the other coordinates in their order. Since $x_p = c + \sum_{i>p} t_i x_i$ on A, ψ is an affine bijection. It preserves the binary order: if $x \neq y$ in A and q is the largest index with $x_q \neq y_q$, then $q \neq p$, because agreement above p forces $x_p = y_p$, so the comparison is decided at a coordinate that ψ keeps. Hence $e_A = e_{\psi^{-1}(0)} \wedge \cdots \wedge e_{\psi^{-1}(7)}$, and with $\sigma = \psi\circ\varphi$ the wedge in question is $\mathrm{sgn}(\sigma)\, e_A$. Every affine bijection of $\mathbb{F}_2^3$ is an even permutation of the 8 points: a nonzero translation is a product of four disjoint transpositions, and $\mathrm{GL}(3,2)$ is generated by the transvections $x \mapsto x + (a\cdot x)b$ with $a\cdot b = 0$, each of which fixes four points and swaps the other four in two pairs. So $\mathrm{sgn}(\sigma) = 1$. For the second statement, $\Lambda^8(\Pi_g)e_A = e_{g(\psi^{-1}(0))} \wedge \cdots \wedge e_{g(\psi^{-1}(7))}$, which is $e_{g(A)}$ by the first statement, as $g \circ \psi^{-1}$ is an affine bijection onto the affine hyperplane $g(A)$. Since g permutes the 30 affine hyperplanes, $\Lambda^8(\Pi_g)w = w$. □

The ordering convention. Relabelling the qubits, reversing the bit order, or listing $\mathbb{F}_2^4$ in Gray-code order replaces the binary order by its transport under a linear bijection L of $\mathbb{F}_2^4$ (for the Gray order, L is the inverse of $k \mapsto k \oplus \lfloor k/2\rfloor$). If A is listed so that $L(a_1) < \cdots < L(a_8)$, then $a_i = L^{-1}\psi'^{-1}(i-1)$ with ψ' the order isomorphism above for $L(A)$, and Lemma 4 with $\varphi = L^{-1}\circ\psi'^{-1}$ shows that the wedge is e_A. So all these conventions give exactly the same w, in particular the same w up to sign. A non-affine order changes the signs of some e_A and destroys the witness (Section 6).

Lemma 5 (Self-duality of RM(1, 3))**.** *Let $\mathcal{R}$ be the set of the 16 affine functions $f(y) = a \cdot y + c$ from $\mathbb{F}_2^3$ to $\mathbb{F}_2$, i.e. the first-order Reed–Muller code* RM(1, 3) *of length 8 [6], and put $\langle f, g\rangle_2 = \sum_y f(y)g(y) \bmod 2$. Then $\mathcal{R} = \mathcal{R}^\perp$, and for every $g\colon \mathbb{F}_2^3 \to \mathbb{F}_2$ the sum $\sum_{f\in\mathcal{R}}(-1)^{\langle f,g\rangle_2}$ equals 16 if $g \in \mathcal{R}$ and 0 otherwise.*

Proof. The support of a nonconstant $f \in \mathcal{R}$ is an affine plane of $\mathbb{F}_2^3$, with 4 points, and two affine planes meet in 0, 2 or 4 points; the constants have support $\emptyset$ or $\mathbb{F}_2^3$. Hence $\langle f, f'\rangle_2 \equiv |\operatorname{supp} f \cap \operatorname{supp} f'| \equiv 0$ for $f, f' \in \mathcal{R}$, i.e. $\mathcal{R} \subseteq \mathcal{R}^\perp$. Since $\mathcal{R}$ is a 4-dimensional subspace of $\mathbb{F}_2^8$, $\mathcal{R}^\perp$ is 4-dimensional as well, and $\mathcal{R} = \mathcal{R}^\perp$. The map $f \mapsto (-1)^{\langle f,g\rangle_2}$ is a character of the group $\mathcal{R}$, trivial exactly when $g \in \mathcal{R}^\perp = \mathcal{R}$, and a character sums to $|\mathcal{R}|$ or 0 over $\mathcal{R}$. ☐

Proposition 6 (Clifford invariance)**.** *For all qubits $j \neq k$, $\Lambda^8(S_j)w = w$, $\Lambda^8(\mathrm{CNOT}_{jk})w = w$ and $\Lambda^8(\sqrt{2}\,H_j)w = 16\,w$. Hence $\Lambda^8(C)w = w$ for every C in the group $\mathcal{C}$ generated by the H_j, S_j and CNOT_{jk}, which contains a representative of every 4-qubit Clifford unitary modulo phases.*

Proof. Phase gate. $\Lambda^8(S_j)e_A = i^{|A\cap\{x_j=1\}|}e_A$, and for an affine hyperplane A the exponent is 0 or 8 if $A = A_{\varepsilon_j,c}$ and 4 otherwise (as in the proof of Lemma 3), so the phase is 1. *CNOT.* $\mathrm{CNOT}_{jk} = \Pi_g$ for the linear bijection $g(x) = x + x_j\varepsilon_k$, and Lemma 4 applies. *Hadamard.* Write $x = (b, y)$ with $b = x_j$ and $y \in \mathbb{F}_2^3$ the other three bits in their order, so that $\sqrt{2}H_je_{(b,y)} = e_{(0,y)} + (-1)^be_{(1,y)}$. Split the hyperplanes $A_{t,c}$ according to t_j. If $t_j = 0$ (14 hyperplanes), then $A = \{(b, y) : y \in A'\}$ for a 4-set $A' \subset \mathbb{F}_2^3$, and $e_A = \pm\bigwedge_{y\in A'}(e_{(0,y)} \wedge e_{(1,y)})$, where the 2-vectors commute. Since $(e_{(0,y)} + e_{(1,y)}) \wedge (e_{(0,y)} - e_{(1,y)}) = -2\,e_{(0,y)} \wedge e_{(1,y)}$, we get $\Lambda^8(\sqrt{2}H_j)e_A = (-2)^4e_A = 16\,e_A$. If $t_j = 1$ (16 hyperplanes), then A is the graph $\Gamma_f = \{(f(y), y) : y \in \mathbb{F}_2^3\}$ of $f(y) = c + t' \cdot y$, where $t' \in \mathbb{F}_2^3$ is t without its j-th coordinate, and f runs over $\mathcal{R}$ bijectively as (t, c) runs over the pairs with $t_j = 1$. As $y \mapsto (f(y), y)$ is an affine bijection $\mathbb{F}_2^3 \to A$, Lemma 4 gives $e_{\Gamma_f} = \bigwedge_y e_{(f(y),y)}$, with y in binary order. Expanding,

$$\Lambda^8(\sqrt{2}H_j)\, e_{\Gamma_f} = \bigwedge_y (e_{(0,y)} + (-1)^{f(y)}e_{(1,y)}) = \sum_{g\colon \mathbb{F}_2^3\to\mathbb{F}_2} (-1)^{\langle f,g\rangle_2} \bigwedge_y e_{(g(y),y)}.$$

Summing over $f \in \mathcal{R}$ and using Lemma 5, the terms with $g \notin \mathcal{R}$ cancel, and by Lemma 4 again $\sum_{f\in\mathcal{R}} \Lambda^8(\sqrt{2}H_j)e_{\Gamma_f} = 16\sum_{g\in\mathcal{R}} e_{\Gamma_g}$. Adding the two parts, $\Lambda^8(\sqrt{2}H_j)w = 16\,w$, that is, $\Lambda^8(H_j)w = w$. The last statement holds because Λ^8 is multiplicative. ☐

Proposition 7 (All Pauli operators)**.** *For every $P \in \mathcal{P}_4$, $\langle w, \Pi_0^P w\rangle = 28$. Hence $\langle w, Mw\rangle = 28 = \frac{14}{15}\|w\|^2$.*

Proof. For each qubit j let C_j be 1, H or HS^3 according as the factor of P on qubit j is I or Z, X, or Y. Since $HXH = Z$ and $S^\dagger YS = X$, the operator $C = \prod_j(C_j)_j \in \mathcal{C}$ satisfies $CPC^\dagger = \pm Z^{s(P)}$, and $s(P) \neq 0$. By Proposition 6, $\Lambda^8(C)w = w$, and $\Lambda^8(C)\Pi_0^P\Lambda^8(C)^{-1} = \Pi_0^{CPC^\dagger} = \Pi_0^{Z^{s(P)}}$. Therefore $\langle w, \Pi_0^P w\rangle = \langle\Lambda^8(C)w, \Pi_0^{Z^{s(P)}}\Lambda^8(C)w\rangle = \langle w, \Pi_0^{Z^{s(P)}}w\rangle = 28$ by Lemma 3. Averaging over the 255 operators P gives the second statement. Only the local gates H_j and S_j were used. ☐

Proposition 8 (Upper bound)**.** *On $\Lambda^8\mathbb{C}^{16}$, $\sum_{P\in\mathcal{P}_4} H_P^2 = 1088 \cdot 1$. Consequently $M \leq \frac{14}{15} \cdot 1$.*

Proof. The 256 Pauli operators $Q \in \mathcal{P}_4 \cup \{I\}$ satisfy $\sum_Q Q \otimes Q = 16\,F$, where F swaps two tensor factors. With $H_Q = \sum_{a=1}^8 Q^{(a)}$ on antisymmetric tensors, where each transposition F_{ab} acts as -1, we get $\sum_Q H_Q^2 = \sum_Q\sum_{a,b} Q^{(a)}Q^{(b)} = 256 \cdot 8 + 16\sum_{a\neq b} F_{ab} = 2048 - 16 \cdot 56 = 1152$, and removing $H_I^2 = 64$ leaves 1088. Every P-charge c lies in $\{-8, \ldots, 8\}$, so $\delta_{c,0} \leq 1 - c^2/64$, i.e. $\Pi_0^P \leq 1 - H_P^2/64$. Averaging over P gives $M \leq 1 - \frac{1088}{64\cdot255} = 1 - \frac{17}{255} = \frac{14}{15}$. ☐

Quadratic Casimirs are also the tool of [2]. Independently, Lemma 10 with $|\lambda| = 8$ ($v = 3$) gives $\|M\| \leq (16 - 2)/15 = 14/15$.

Proof of Theorem 1. By Proposition 8, $Q = \frac{14}{15} - M$ is positive semidefinite, and by Proposition 7, $\langle w, Qw\rangle = 0$. Hence $\|Q^{1/2}w\|^2 = 0$ and $Qw = 0$, i.e. $Mw = \frac{14}{15}w$ and $\|M\| = \frac{14}{15}$. As $\Lambda^8\mathbb{C}^{16}$ is a nontrivial irreducible representation, Lemma 2(c) gives $\Delta_4 \le 1 - \frac{14}{15} = \frac{1}{15} = \frac{17}{255} < \frac{26}{255}$. At $n = 4$ the lower bound of [1, Thm. 3.16] is $(4^4 + 16)/(16 \cdot 255) = 272/4080 = 1/15$, so $\Delta_4 = 1/15$. $\square$

Corollary 9 (Eigenfunction)**.** *For an* 8*-subset* $B_0 \subset \mathbb{F}_2^4$ *let* $f(U) = \sum_A \det U[A, B_0]$*, summed over the* 30 *affine hyperplanes* A *of* $\mathbb{F}_2^4$*, with rows and columns in ascending binary order. Then* f *is a nonzero element of* $L_0^2(\mathrm{SU}(16))$*,* $K_4 f = \frac{14}{15}f$*, and* $f(\omega U) = -f(U)$ *for* $\omega = e^{i\pi/8}$*; for* $B_0 = \{0, 1, \ldots, 7\}$*,* $f(1) = 1$*. For every* U *and every integer* $N \ge 9$*,*

$$(K_4 f)(U) = \frac{1}{255\,N} \sum_{P \in \mathcal{P}_4} \sum_{m=0}^{N-1} f(e^{2\pi i m P/N}U). \tag{6}$$

Proof. By the Cauchy–Binet formula, $f(U) = \langle w, \Lambda^8(U)e_{B_0}\rangle$, a matrix coefficient of the nontrivial irreducible representation $\Lambda^8\mathbb{C}^{16}$. It has zero Haar mean by Schur orthogonality, and it is nonzero because the vectors $\Lambda^8(U)e_{B_0}$ span $\Lambda^8\mathbb{C}^{16}$. Lemma 2(b) and Theorem 1 give $K_4 f = \frac{14}{15}f$, and $\Lambda^8(\omega 1) = \omega^8 = -1$. As $\{0, \ldots, 7\} = A_{\varepsilon_3,0}$, $f(1) = \langle w, e_{B_0}\rangle = 1$. Finally, $e^{i\theta P} = \cos\theta \cdot 1 + i\sin\theta \cdot P$, so each 8×8 minor of $e^{i\theta P}U$ is a trigonometric polynomial of degree at most 8 in θ, and for $N \ge 9$ its average over the points $2\pi m/N$ equals its average over $[0, 2\pi)$. $\square$

4 A selection rule

The upper half of Theorem 1 is an instance of a bound that holds for every n and depends only on $|\lambda|$.

Lemma 10 (Selection rule)**.** *Let* $n \ge 2$*,* $d = 2^n$*, and let* λ *be a nontrivial irreducible representation of* $\mathrm{SU}(d)$*. Let* $v \in \{0, \ldots, n\}$ *be the* 2*-adic valuation of* $|\lambda| \in \mathbb{Z}/d\mathbb{Z}$*, with* $v = n$ *when* $|\lambda| = 0$*. If* $v < n$*, then*

$$\|M_\lambda\| \le \frac{2^n - 2^{n-v}}{2^n - 1}.$$

In particular $M_\lambda = 0$ *when* $|\lambda|$ *is odd.*

Proof. Fix a polynomial representative of degree $k \equiv |\lambda| \pmod{d}$ and a basis of weight vectors (Section 2). Since $v < n$, $2^{v+1} \nmid k$.

(a) Charges. H_{Z^s} acts on the weight space of weight μ as the Walsh coefficient $c_s(\mu) = \sum_x (-1)^{s\cdot x}\mu_x$, and $c_0(\mu) = k$. Let $\mathcal{N}(\mu) = \{s \ne 0 : c_s(\mu) = 0\}$ be the set of neutral Z-type directions.

(b) Divisibility. Let $W \subseteq \mathbb{F}_2^n$ be a subspace of dimension m with $W \setminus \{0\} \subseteq \mathcal{N}(\mu)$. For every $y \in \mathbb{F}_2^n$,

$$2^m \sum_{x \in y + W^\perp} \mu_x = \sum_x \mu_x \sum_{s \in W} (-1)^{s\cdot(x+y)} = \sum_{s \in W} (-1)^{s\cdot y} c_s(\mu) = c_0(\mu) = k,$$

so $2^m \mid k$. Hence $\mathcal{N}(\mu) \cup \{0\}$ contains no subspace of dimension $v + 1$.

(c) Extremal count. If $\mathcal{N} \subseteq \mathbb{F}_2^n \setminus \{0\}$ and $\mathcal{N} \cup \{0\}$ contains no subspace of dimension $v + 1$, where $0 \le v < n$, then $|\mathcal{N}| \le 2^n - 2^{n-v}$. This is the case $q = 2$ of a theorem of Bose and Burton [3]; we prove it by induction on v. For $v = 0$, $\mathcal{N} = \emptyset$. Let $v \ge 1$ and suppose $|\mathcal{N}| > 2^n - 2^{n-v}$, so that $B = \mathbb{F}_2^n \setminus (\mathcal{N} \cup \{0\})$ has $|B| < 2^{n-v} - 1$. Pick $z \in \mathcal{N}$ and let $\pi\colon \mathbb{F}_2^n \to \mathbb{F}_2^n/\langle z\rangle \cong \mathbb{F}_2^{n-1}$ be the quotient map. Then $\pi(B)$ avoids 0, as $0, z \notin B$, and $\mathcal{N}' = \mathbb{F}_2^{n-1} \setminus (\pi(B) \cup \{0\})$ has $|\mathcal{N}'| > 2^{n-1} - 2^{(n-1)-(v-1)}$. By induction $\mathcal{N}' \cup \{0\}$ contains a v-dimensional subspace W', and $W = \pi^{-1}(W')$ is $(v+1)$-dimensional with $W \setminus \{0\} \subseteq \mathcal{N}$: $z \in \mathcal{N}$, and any other $x \in W \setminus \{0\}$ has $\pi(x) \in W' \setminus \{0\} \subseteq \mathcal{N}'$, so $\pi(x) \notin \pi(B)$, $x \notin B$ and $x \in \mathcal{N}$. This is a contradiction. With (b), $|\mathcal{N}(\mu)| \le 2^n - 2^{n-v}$ for every weight μ.

(d) Averaging over stabiliser groups. For $\ell = (a,b) \in \mathbb{F}_2^{2n} \setminus \{0\}$ let $P_\ell \in \mathcal{P}_n$ be proportional to $X^a Z^b$; P_ℓ and $P_{\ell'}$ commute if and only if $a \cdot b' + a' \cdot b = 0$. For a Lagrangian subspace $L \subset \mathbb{F}_2^{2n}$ (isotropic, of dimension n) the P_ℓ with $\ell \in L \setminus \{0\}$ commute, and if $g_1, \dots, g_n$ correspond to a basis of L, each joint eigenspace of $(g_1, \dots, g_n)$ is one-dimensional, since its projector $2^{-n} \prod_i (1 \pm g_i)$ has trace 1. Label the joint eigenvectors ψ_x, $x \in \mathbb{F}_2^n$, so that $g_i \psi_x = (-1)^{x_i} \psi_x$. A unitary $V \in \mathrm{SU}(d)$ with $V\psi_x \propto e_x$ then satisfies $V P_\ell V^\dagger = \pm Z^{\sigma(\ell)}$, where $\sigma\colon L \to \mathbb{F}_2^n$ takes coordinates in the basis. So $\sum_{\ell \in L \setminus \{0\}} \Pi_0^{P_\ell}$ is unitarily equivalent to $\sum_{s \neq 0} \Pi_0^{Z^s}$, which acts on the weight space of μ as $|\mathcal{N}(\mu)| \leq 2^n - 2^{n-v}$. The Lagrangians containing a given $\ell \neq 0$ correspond to the Lagrangians of the $(2n-2)$-dimensional symplectic space $\ell^\perp / \langle \ell \rangle$, so their number does not depend on ℓ, and $M_\lambda = \mathbb{E}_L\, (2^n - 1)^{-1} \sum_{\ell \in L \setminus \{0\}} \Pi_0^{P_\ell}$ with L uniform. This gives the bound. For odd k all $c_s(\mu)$ are odd, so $\mathcal{N}(\mu) = \emptyset$ and $M_\lambda = 0$. □

Corollary 11. *Let $n \geq 3$ and $\Delta_n^\star = 2^n(2^n - 3)/(8(4^n - 1))$, so that $1 - \Delta_n^\star = (7 \cdot 4^n + 3 \cdot 2^n - 8)/(8(4^n - 1))$. Every nontrivial irreducible representation λ of $\mathrm{SU}(2^n)$ with $8 \nmid |\lambda|$ satisfies*

$$\|M_\lambda\| \leq \frac{3 \cdot 2^{n-2}}{2^n - 1} = 1 - \Delta_n^\star - \frac{2^n(2^n - 3) - 8}{8(4^n - 1)} < 1 - \Delta_n^\star.$$

For $n \geq 4$ and $|\lambda| \equiv 8 \pmod{16}$ the bound of Lemma 10 is $(2^n - 2^{n-3})/(2^n - 1) = 1 - \Delta_n^\star + (4 \cdot 2^n + 8)/(8(4^n - 1))$, which exceeds $1 - \Delta_n^\star$.

Proof. If $8 \nmid |\lambda|$, then $v \leq 2 < n$, and the bound of Lemma 10 increases with v. The identities follow from $3 \cdot 2^{n-2}/(2^n - 1) = (6 \cdot 4^n + 6 \cdot 2^n)/(8(4^n - 1))$ and $(2^n - 2^{n-3})/(2^n - 1) = (7 \cdot 4^n + 7 \cdot 2^n)/(8(4^n - 1))$, and $2^n(2^n - 3) - 8 \geq 32$ for $n \geq 3$. □

So for $n \geq 3$ a representation that violates Eq. (2) must have $8 \mid |\lambda|$; at $n = 4$ it is balanced or has $|\lambda| = 8$, and Theorem 1 shows that $\Lambda^8 \mathbb{C}^{16}$, with $|\lambda| = 8$, does violate it.

Remark 12 (Sharpness). The bound of Lemma 10 is attained in at least three cases. (i) $\Lambda^2 \mathbb{C}^d$ ($v = 1$), every $n \geq 2$: the charges lie in $\{0, \pm 2\}$, so $\Pi_0^P = 1 - H_P^2/4$, and the computation of Proposition 8 with two factors gives $\sum_{P \in \mathcal{P}_n} H_P^2 = 2d^2 - 2d - 4$, hence $M = \frac{d}{2(d-1)} 1 = \frac{2^{n-1}}{2^n - 1} 1$. (ii) $\Lambda^4 \mathbb{C}^8$ at $n = 3$ ($v = 2$): the same Casimir computation gives $\|M\| \leq 6/7$, and equality is certified by the program of Section 6 and was found numerically in the independent review. (iii) $\Lambda^8 \mathbb{C}^{16}$ at $n = 4$ ($v = 3$), by Theorem 1.

5 Discussion

An unbalanced mode. $\Lambda^8 \mathbb{C}^{16}$ has $|\lambda| = 8 \not\equiv 0 \pmod{16}$: the centre of $\mathrm{SU}(16)$ acts on it by $\omega^8 = -1$, the eigenfunction of Corollary 9 changes sign under $U \mapsto \omega U$, and λ occurs in no $U^{\otimes t} \otimes \bar{U}^{\otimes t}$. Theorem 1 concerns Δ_n exactly as defined in (1), which is also the gap of [1, Def. 1.1]; for unitary designs only the balanced representations, i.e. functions on $\mathrm{PU}(2^n)$, matter [1]. The balanced gap at $n = 4$, $1 - \sup \|M_\lambda\|$ over the nontrivial λ with $|\lambda| = 0$, lies between $1/15$ and $26/255$ by (3); it is not settled here.

Why $n = 4$. First, the Hadamard step of Proposition 6 uses the self-duality of the affine functions on $\mathbb{F}_2^{n-1}$, and $\mathrm{RM}(1, m)^\perp = \mathrm{RM}(m - 2, m)$ [6] equals $\mathrm{RM}(1, m)$ only for $m = 3$. Second, for the middle exterior power $\Lambda^{2^{n-1}} \mathbb{C}^{2^n}$ and the analogous sum w_n over the $2(2^n - 1)$ affine hyperplanes of $\mathbb{F}_2^n$, the Casimir computation of Proposition 8 gives $\|M\| \leq 1 - 1/(2^n - 1)$, and the count of Lemma 3 gives $\langle w_n, \Pi_0^{Z^s} w_n \rangle = (1 - 1/(2^n - 1)) \|w_n\|^2$ for every Z-type Pauli operator. But $1/(2^n - 1) - (4^n + 16)/(16(4^n - 1)) = 2^n(16 - 2^n)/(16(4^n - 1))$ vanishes only at $n = 4$. For $n \geq 5$ it is negative, so by [1, Thm. 3.16] the Z-type value of w_n cannot extend to all Pauli operators.

What remains open. Δ_n for $n \geq 5$ is open: Eq. (2) may hold or fail there, and by Corollary 11 a violating representation must have $8 \mid |\lambda|$. So is the balanced gap at $n = 4$. We do not claim that $\Lambda^8\mathbb{C}^{16}$ is the only representation with $\|M_\lambda\| = 14/15$; on it, a Lanczos computation in the review indicates that $14/15$ is a simple eigenvalue, followed by $193/255$.

Relation to prior work. On 28 September 2026 the arXiv listing of [1] showed only version 1 (23 July 2026). It conjectures the opposite of Theorem 1, names a balanced representation as tight, and supports this with $\tau_{4,4}$ for $n \leq 3$. The independent review found in it neither w, nor the value $\Delta_4 = 1/15$, nor the selection rule, and no exterior power as a candidate for the gap. Haah, Liu and Tan treat the balanced moment operators that govern unitary designs [2]; for their work only the abstract was consulted. We did not search systematically for later work citing [1].

6 Verification

The file `check_pauli_rotation_gap.py` is standalone. It uses only the Python standard library and no network, takes no input, and works in exact integer, Gaussian-integer and rational arithmetic except for the floating-point check [E1]. It prints 22 tagged checks and a verdict line, fixed in `expected_stdout.txt`. [B1] checks the Baer–Haah values and that the lower bound in (3) equals $1/(2^n - 1)$ only at $n = 4$ ($n \leq 64$). [W1]–[W2] build the 30 hyperplanes and w and check Lemma 3 for all 15 Z-type Pauli operators. [W3]–[W4] check the order lemma, including the images of every e_A under all of $\mathrm{GL}(4, 2)$ and all translations, and the bit-reversed and Gray-code orders. [W5]–[W6] check the self-duality of $\mathrm{RM}(1, 3)$ and Proposition 6 for all 20 generators, by exact expansion in the exterior algebra with the wedge signs of the binary order. [W7]–[W8] check the local Clifford reduction, $\langle w, \Pi_0^P w\rangle = 28$ for all 255 operators P, and the Rayleigh quotient $14/15 > 229/255$. [W9] is a route without Clifford invariance: writing $P = i^{a\cdot b} X^a Z^b$, it works with the sparse integer matrices $H'_P = i^{-a\cdot b} H_P$ on the 12870-dimensional space (so $H_P^2 = (-1)^{a\cdot b} H_P'^2$), checks their signs against a direct wedge expansion of $d\rho(P)$, computes $t_P = \prod_{c\in\{2,4,6,8\}} (H_P^2 - c^2)\, w$, checks $H_P t_P = 0$, so that $t_P = 147456\, \Pi_0^P w$, and verifies $\sum_P t_P = 147456 \cdot 238\, w$, i.e. $Mw = \frac{14}{15} w$ exactly. [W10] checks $\sum_P H_P^2 e_B = 1088\, e_B$ on the hyperplane wedges and on seeded random basis vectors. [S1]–[S2] check the divisibility step and, exhaustively for $n = 3, 4$, the extremal count of Lemma 10; [S3] checks the equality cases of Remark 12 ($\Lambda^2\mathbb{C}^d$ for $2 \leq n \leq 4$; for $\Lambda^4\mathbb{C}^8$, $\frac{6}{7} \cdot 1 - M$ is singular and positive semidefinite by exact $LDL^\top$); [S4] checks the inequalities of Corollary 11 for $3 \leq n \leq 64$. The positive controls [C1]–[C2] apply the same wedge-sign and charge conventions to the traceless part of the Λ^4 code projector in $\mathrm{End}(\Lambda^4\mathbb{C}^d)$, a vector of the balanced fourth-moment sector, and recover $58/63 = 1 - \Delta_3$ at $n = 3$ and $229/255 = 1 - 26/255$ at $n = 4$. The negative controls [N1]–[N3] show that random sign patterns on the 30 wedges and non-affine orders of $\mathbb{F}_2^4$ give Rayleigh quotients below $229/255$, and that every 8-subset of $\mathbb{F}_2^4$ other than the hyperplanes has at most 11, not 14, vanishing Z-type charges. [E1] evaluates (6) with $N = 17$ in floating point at three pseudo-random unitaries, and [T] states the conclusion of Theorem 1.

The program runs in about 10 s (CPython 3.13 on a laptop; about 11 s in an isolated container with CPython 3.12 and no network); both runs, repeated twice, reproduce `expected_stdout.txt` byte for byte, as recorded in `reproduction.json`. The harness `mutate.py` applies 32 single-point mutations spread over all tags, recorded in `MUTATION_REPORT.txt`; each makes the program exit with a nonzero status, and none survives. Reproduction establishes only that the program computes what it prints; the reduction, the order lemma, the Hadamard step and the selection rule rest on the written argument.

AI assistance and review status

AI tools were used to develop the argument, to write the verification program, to assist with exposition and to conduct internal adversarial reviews. The manuscript follows the evidence-tracking guidance of Scientific Agent Skills [8]. These activities are disclosed as AI assistance, not as external refereeing or formal proof verification.

An adversarial review was performed by AI in an isolated context, with its own code. It confirmed from [1, Def. 1.1] that unbalanced representations are within the scope of Conjecture 3.48. It recomputed $\langle w, \Pi_0^P w\rangle = 28$ for all 255 Pauli operators in two independent ways (from 8×8 minors of $e^{i\theta P}$ on an exact θ-grid, and from its own derivation representation), confirmed the Clifford invariance by Cauchy–Binet expansion over all 12870 coordinates, and found by Lanczos iteration that the two largest eigenvalues of M are 14/15 and 193/255. In sign attacks, 3000 random sign patterns gave Rayleigh quotients at most 0.785 and three random basis orders 0.49–0.65, while bit-reversed and Gray-code orders gave the same w up to sign. It evaluated the eigenfunction of Corollary 9 at three Haar-random unitaries, checked Lemma 10 line by line, and graded the argument correct, aligned with the recorded question, a complete answer, and new relative to the sources read. Its requested edits (define the order behind "ascending", call $\Lambda^8\mathbb{C}^{16}$ *a* bottleneck, and state the eigenfunction) are incorporated. A separate check with separate code computed $\langle w, \Pi_0^P w\rangle/\langle w, w\rangle$ in the eigenbasis of each of the 255 Pauli operators from 8×8 minors, without Clifford invariance, and obtained 14/15 each time. Earlier internal rounds studied the fourth-moment sector; the argument here does not rely on them. **No independent human expert has examined this argument, and no novelty certification by a human is recorded.** The problem record [7] was listed as unsolved when this manuscript was prepared.

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