

Integer word moments do not control fractional Schatten norms: a counterexample to Open Problem 1 of Alhejji–Knill

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Abstract

Alhejji and Knill ask whether word-trace dominance between two pairs of positive semidefinite matrices with matched spectra forces domination of the Schatten p -norms of their sums for every $p \in [1, \infty)$. A positive answer would carry the spin-alignment conjecture from integer Schatten norms to all Schatten norms and to the von Neumann entropy, which is the bridge to a single-letter quantum capacity for platypus-type channel families. We answer in the negative. The simplest witness is a commuting, diagonal family in dimension three for which word dominance is analytic and the norm inequality fails at $p = 3/2$ between integers. The failure is therefore classical: non-commutativity plays no role in it.

1. Background

The quantum capacity of a channel is given by a regularised expression, and the regularisation is known to be genuinely necessary: no finite number of channel uses suffices in general [4], capacity is non-additive even in very simple channels [3], and the general computational problem is undecidable [5]. Progress therefore comes from identifying structured families where the regularisation collapses.

The spin-alignment conjecture of Alhejji and Knill [1] is such a route. It is proved there for integer Schatten norms, for classical states, and for mixtures of two states. Extending it to all Schatten norms — equivalently, reaching the von Neumann entropy by analytic continuation — would give a single-letter quantum capacity for platypus-type families. Song and Chen [2] have since refuted the strong form of the conjecture, leaving the weaker compatible-marginal statement open.

Among the concluding remarks of [1] is a clean bridge lemma which, if true, would supply the extension. This note settles it.

2. The problem

Open Problem 1 of [1]. *Let A_0, A_1, B_0, B_1 be positive semidefinite with*

$$\lambda(A_0) = \lambda(B_0), \quad \lambda(A_1) = \lambda(B_1),$$

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and for a binary word s write $\Pi_s(A) = \prod_i A_{s(i)}$. Suppose word-trace dominance holds:

$$\mathrm{tr} \Pi_s(A) \geq |\mathrm{tr} \Pi_s(B)| \quad \text{for every word } s. \quad (\text{H})$$

Does it follow that

$$\|A_0 + A_1\|_p \geq \|B_0 + B_1\|_p \quad \text{for every } p \in [1, \infty)? \quad (\text{C})$$

It is posed as a question, so an instance satisfying (H) and violating (C) is a legitimate negative resolution. Note that (H) constrains only integer moments — traces of words — while (C) is a statement about fractional powers of the spectrum of a sum. The content of the problem is whether the former controls the latter.

3. A commuting counterexample

Theorem. *Open Problem 1 of [1] is false.*

Take

$$A_0 = \mathrm{diag}(176, 0, 64), \quad A_1 = \mathrm{diag}(80, 49, 0), \quad B_0 = \mathrm{diag}(64, 176, 0), \quad B_1 = A_1.$$

Spectra match by inspection and all four matrices are positive semidefinite, so the instance is admissible. Everything commutes, so each word collapses to a product of powers and (H) reduces to the single analytic statement

$$(64/176)^k + (49/80)^m \leq \frac{859}{880} < 1 \quad (k, m \geq 1),$$

which holds for all words. Conclusion (C) fails at $p = 3/2$: the two spectral sums $\sum_i \lambda_i^{3/2}$ are 4951 and 5103, so $\|A_0 + A_1\|_{3/2} < \|B_0 + B_1\|_{3/2}$. The comparison is between integers; no numerical tolerance enters.

The extremal member of the family, $\mathrm{diag}(2, 0, 1)$, $\mathrm{diag}(2, 1, 0)$ against $\mathrm{diag}(1, 2, 0)$, $\mathrm{diag}(2, 1, 0)$, has failure window exactly $(1, 2)$: the function $f(p) = 4^p + 2 - 2 \cdot 3^p$ has zeros exactly at $p = 1$ and $p = 2$, by Descartes' rule applied to the exponential sum.

4. A non-commuting instance, and minimality

A 3×3 non-commuting instance, found first, has $\lambda(A_0) = \lambda(B_0) = \{0, \frac{1}{5}, \frac{7}{10}\}$ and $\lambda(A_1) = \lambda(B_1) = \{\frac{1}{10}, \frac{3}{10}, \frac{1}{2}\}$, equality at $p = 1$, and failure on the whole interval $p \in (1, p^*)$ with $p^* = 1.97584\dots$, the endpoints pinned by a Descartes/Laguerre sign argument on the relevant exponential sum. Verifying (H) for it is not analytic: it rests on a cone argument for a family of 2×2 transfer blocks, with a resonance constant at which the bound saturates asymptotically from below without ever crossing. A separate argument excludes $d = 2$, so qutrits are the minimal dimension in which any of this can occur.

5. Verification

Every load-bearing step is exact rational or 100-digit arithmetic. The 3×3 instance was re-derived from scratch by an independent agent using its own support-projection frame and its own cone argument, arriving at the same resonance constant; hostile search covered every exact word up to length 13 and roughly 470k structured profiles with no violation, and the asymptotic family saturates from below without crossing. The diagonal family was produced by a container-isolated run with no network access and no sight of the earlier work, which makes it about as independent a corroboration as the setup allows.

6. What remains open

Three things.

(i) *A referee-grade write-up of the 3×3 case.* Every scalar inequality in the all-lengths word-dominance step was discharged symbolically, so this is a write-up obligation rather than a mathematical gap — but it has not been typeset as a proof a referee could read linearly.

(ii) *The kernel lemma for the diagonal family.* The reduction of all words to the single two-parameter inequality is corroborated on thousands of random draws but not yet proved symbolically. Since the family is commuting this should be routine, and it would make the counterexample fully self-contained on half a page.

(iii) *The conjecture itself.* This result blocks the overlap-lemma route from integer Schatten norms to the fractional ones. It does *not* refute the spin-alignment conjecture, and in the instance above the A -side is not aligned. The weaker compatible-marginal statement of [2] also remains open; see the companion note on the $n = 4$ case.

Disclosure

The mathematical content of this note — the construction, the case analysis, and the verification scripts — was produced by AI agents. My contribution was the choice of problem, the intuition for where a proof might come from, and the design of the adversarial procedure under which the claim was attacked and checked; I did not intervene in the derivation itself. Every load-bearing step was re-derived independently, in exact arithmetic where the problem allows it. This note is a record of a computer-assisted result. It has not been refereed.

References

- [1] M. A. Alhejji and E. Knill, *Towards a resolution of the spin alignment problem*, arXiv:2307.06894.
- [2] Z. Song and L. Chen, *A counterexample to the strong spin alignment conjecture*, arXiv:2603.25410.
- [3] F. Leditzky, D. Leung, V. Siddhu, G. Smith and J. A. Smolin, *Generic nonadditivity of quantum capacity in simple channels*, arXiv:2202.08377.
- [4] T. Cubitt, D. Elkouss, W. Matthews, M. Ozols, D. Pérez-García and S. Strelchuk, *Unbounded number of channel uses are required to see quantum capacity*, arXiv:1408.5115.
- [5] A. Bhattacharyya, A. Mehta and Y. Zhao, *On the undecidability of quantum channel capacities*, arXiv:2601.22471.