

Gaussian entanglement of formation for one mode versus finitely many modes

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22 October 2026 (version 1)

Abstract

We prove that the entanglement of formation of every finite-energy Gaussian state with one mode on one side and finitely many modes on the other equals its Gaussian restriction. The result includes arbitrary position–momentum cross correlations, nonsymmetric states, physical boundary covariances, and convex roofs defined using arbitrary pure-state probability measures. The lower bound uses Adesso’s affine two-mode entanglement–EPR inequality. The additional step is a covariance optimization argument: strict physicality supplies a constraint qualification on the manifold of pure covariances, and the corresponding positive multiplier has rank two. It therefore produces an EPR witness supported on the single entangled pair of an optimal pure seed. Complementary slackness makes the retained two-mode state saturate the imported affine bound, even when the mixed state remains correlated with the other modes. Local additive noise and compactness extend the equality to every physical boundary.

1 Statement and relation to the two-mode problem

A Gaussian restriction of the entanglement convex roof is useful only if one understands when non-Gaussian decompositions can improve it. The Gaussian covariance minimization is available for arbitrary numbers of modes [1, Proposition 1]. The unrestricted roof is a separate problem. The generalized EPR approach has a long two-mode history [3]. Adesso’s affine inequality establishes the required two-mode lower bound and its consequences for two-mode and bisymmetric Gaussian states [2, Theorem 1].

Here we show that a Gaussian optimum for a general one-versus-many state always supplies a saturating two-mode witness in the strictly physical case. This conclusion does not follow by putting the *mixed* state into a two-mode tensor-product form. Instead, we put an optimal *pure covariance below it* into Schmidt form and examine its stationarity multiplier. A boundary limit then proves the full theorem.

Theorem 1.1. *Let ρ_{AB} be a Gaussian state with finite mean energy, with A one bosonic mode and B any finite number $m \geq 1$ of bosonic modes. Define E_F by the infimum of average pure-state entanglement over all pure-state probability measures with barycenter ρ_{AB} , and define E_F^G by restricting those measures to pure Gaussian states. Then*

$$E_F(\rho_{AB}) = E_F^G(\rho_{AB}).$$

No permutation symmetry, position–momentum block structure, or strict uncertainty inequality is assumed.

In particular, the theorem covers a nonsymmetric three-mode state with one Alice mode and two Bob modes. It makes no assertion for a general bipartition in which both parties have more than one mode. The two-mode affine inequality is an imported theorem, stated precisely below; the new argument concerns the covariance multiplier and its support.

2 Covariance and convex-roof conventions

Order quadratures by modes:

$$R = (Q_A, P_A, Q_{B_1}, P_{B_1}, \dots), \quad [R_j, R_k] = iJ_{jk}, \quad J = \bigoplus_{j=0}^m J_2, \quad J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We use the doubled covariance

$$V_{jk} = \text{Tr } \rho \{R_j - \langle R_j \rangle, R_k - \langle R_k \rangle\}.$$

Thus the vacuum covariance is I , physicality is $V + iJ \geq 0$, and a pure Gaussian covariance Γ obeys $\Gamma J \Gamma = J$. First moments can be removed by local displacements. All logarithms and entropies in this paper are natural.

For a pure state ψ , let $E(\psi) = S(\text{Tr}_B |\psi\rangle\langle\psi|)$. A probability measure μ on pure states is admissible if its barycenter is ρ ; the cost is $\int E(\psi) d\mu(\psi)$. Finite mean energy makes this integral finite for every admissible measure. Indeed, the mean component energy is the energy of ρ by Tonelli's theorem, while

$$S(\psi_A) \leq g(N_A(\psi)), \quad g(x) = (x+1) \log(x+1) - x \log x,$$

and concavity of g bounds the average by $g(N_A(\rho))$. These statements also justify the integrations used below.

Proposition 2.1 (Gaussian covariance roof). *For a Gaussian state with covariance V ,*

$$E_F^G(V) = \min_{\substack{\Gamma > 0, \Gamma J \Gamma = J \\ \Gamma \leq V}} g\left(\frac{\sqrt{\det \Gamma_A} - 1}{2}\right). \quad (1)$$

This identity holds for arbitrary Gaussian pure-state probability measures, and the minimum is attained.

Proof. The covariance characterization is inherited from Ref. [1, Proposition 1, Eq. (9)]. We give the probability-measure and attainment details.

Consider a pure Gaussian decomposition of the Gaussian state. Measurement of a fixed quadrature produces a positive mixture of one-dimensional normal distributions, whose mixture is the target normal distribution. Suppose a positive-measure set of components had variance at least $\epsilon > 0$ above the target variance. Restrict further to a positive-measure subset on which the component means have modulus at most a finite M . The moment-generating function would then be bounded below by a positive constant times

$$\exp\left(\frac{\sigma^2 + \epsilon}{2} t^2 - M|t|\right),$$

where σ^2 is the target variance. This contradicts its Gaussian moment-generating function as $|t| \rightarrow \infty$. The argument is legitimate even before assuming component moment-generating integrability, since the integrand is nonnegative and the target expectation is finite. Thus each fixed quadrature's component variance is at most the target's almost everywhere.

Apply this to a countable dense set of rational directions, intersect their full-measure sets, and use continuity of quadratic forms. Almost every component covariance satisfies $\Gamma \leq V$, so its entropy is at least the right-hand infimum in (1). Conversely, for any feasible pure Γ , displace its Gaussian state by a centered Gaussian distribution of covariance $(V - \Gamma)/2$. This is allowed also for singular $V - \Gamma$, by a measure on its range. The resulting barycenter has covariance V and all components have the same entanglement. A one-mode covariance has symplectic eigenvalue $\sqrt{\det \Gamma_A}$, giving the displayed entropy.

Williamson decomposition supplies at least one pure $\Gamma \leq V$. The feasible set is compact: its largest eigenvalue is at most $\|V\|$, and purity gives

$$\Gamma^{-1} = J^T \Gamma J,$$

so its smallest eigenvalue is at least $1/\|V\|$. Closure preserves positivity and purity. The objective is continuous, including at symplectic eigenvalue one, hence the infimum is attained. $\square$

3 The imported affine two-mode bound

On a pair of modes, let a, b be the annihilation operators and set

$$D_t = (ta - b^\dagger)^\dagger (ta - b^\dagger), \quad 0 < t \leq 1.$$

For $0 < q < t$, define

$$d_t(q) = \frac{(1 - tq)^2}{1 - q^2}, \quad E(q) = g\left(\frac{q^2}{1 - q^2}\right), \quad \lambda_{q,t} = \frac{-2q \log q}{(t - q)(1 - tq)} > 0. \quad (2)$$

Theorem 3.1 (Adesso's affine inequality). *For a finite-energy two-mode state ρ and $0 < q < t \leq 1$,*

$$E_F(\rho) \geq E(q) - \lambda_{q,t}(\text{Tr}(\rho D_t) - d_t(q)). \quad (3)$$

The pure-state inequality is Theorem 1, Eqs. (8)–(9), of Ref. [2]; the limiting and integration interface is discussed there in Supplement S2. We use that theorem as an external input, not as a result proved here. To align its mixed-state consequence with our measure convention, note that $D_t \geq 0$ and $D_t \leq c_t(N_A + N_B + 1)$ as quadratic forms for a finite c_t . For any admissible decomposition of a finite-energy barycenter, component energies are finite almost everywhere, and both the EPR expectation and entropy are integrable. Integrating the pure-state affine inequality and taking the infimum proves (3) without assuming a finite optimal ensemble.

For a centered two-mode state with doubled covariance V_R , put

$$K_t = \begin{pmatrix} t^2 I_2 & -tZ \\ -tZ & I_2 \end{pmatrix}, \quad Z = \text{diag}(1, -1).$$

The conversion from the vacuum- $I/2$ convention in Ref. [2] to ours is

$$\text{Tr}(\rho D_t) = \frac{1}{4} \text{Tr}(K_t V_R) + \frac{1 - t^2}{2}. \quad (4)$$

The constant is essential: D_t contains $bb^\dagger$, whereas its first diagonal term is $t^2 a^\dagger a$. For a two-mode squeezed vacuum with $q = \tanh r$, this expectation equals $d_t(q)$.

4 An optimal pure seed and a strict feasible tangent

It suffices to minimize

$$a(\Gamma) = \sqrt{\det \Gamma_A}$$

over the feasible set in (1), since $g((a-1)/2)$ is increasing for $a \geq 1$. Choose a minimizer Γ_* . If $a(\Gamma_*) = 1$, its Alice marginal is pure, so the pure seed is a product across $A : B$. The displacement ensemble in Proposition 2.1 is separable and both roofs are zero. We henceforth assume $a > 1$, and initially impose strict physicality:

$$V + iJ > 0.$$

Apply local symplectic changes to put the pure covariance in Gaussian Schmidt form. There is one entangled pair, on A and a mode D of Bob, and the other Bob modes are vacuum spectators. Writing

$$q = \tanh r, \quad c = \cosh r, \quad s = \sinh r, \quad 0 < q < 1,$$

we have

$$\Gamma_* = SS^T, \quad S_{\text{pair}} = \begin{pmatrix} cI_2 & sZ \\ sZ & cI_2 \end{pmatrix}, \quad a = c^2 + s^2 = \frac{1+q^2}{1-q^2}, \quad (5)$$

where S acts as S_{pair} on A, D and as the identity elsewhere.

For completeness, this one-mode pure-state reduction can be seen directly. First make $\Gamma_A = aI_2$ by an Alice symplectic change. Writing its cross block as C_{AB} , purity gives

$$C_{AB}J_B C_{AB}^T = -(a^2 - 1)J_2.$$

Its two rows therefore span a nondegenerate symplectic plane on Bob. Extend this plane to a symplectic basis, so that $C_{AB} = (\sqrt{a^2 - 1}Z, 0)$. The off-diagonal block of the purity equation forces the retained Bob covariance to be aI_2 , and its correlations with the spectators to vanish. The spectator covariance is pure and can be changed to the identity by a Bob symplectic transformation. This is the standard pure Gaussian mode decomposition used with the covariance roof [1].

Now whiten algebraically by S^{-1} :

$$V' = S^{-1}VS^{-T} = I + D, \quad D \geq 0.$$

This whitening need not be local; it is a coordinate device for the optimization, not a physical operation on the input. Because S is symplectic, $V' + iJ > 0$. Let $K = \ker D$, and let P_K be its real orthogonal projector.

Lemma 4.1. $K \cap JK = \{0\}$. The symmetric matrix

$$X_0 = JP_K J^T - P_K$$

satisfies $X_0J + JX_0 = 0$, and its compression to K is strictly negative definite if $K \neq \{0\}$.

Proof. If $v, Jv \in K$ with $v \neq 0$, the nonzero vector $z = v - iJv$ satisfies both $Dz = 0$ and $(I + iJ)z = 0$, contradicting $V' + iJ > 0$. This proves transversality. Direct multiplication gives the tangent identity for X_0 . For $v \in K \setminus \{0\}$,

$$v^T X_0 v = \|P_K J^T v\|^2 - \|v\|^2 < 0,$$

because equality would imply $J^T v \in K$, contrary to transversality. $\square$

The tangent space of real symmetric pure covariances at I is

$$\mathcal{T} = \{X = X^T : XJ + JX = 0\}.$$

Every $X \in \mathcal{T}$ is realized by the pure curve $\Gamma'(t) = e^{tX}$: $e^{tX/2}$ is symplectic. The Frobenius-orthogonal complement $\mathcal{T}^\perp$ is the space of symmetric matrices fixed by $X \mapsto JXJ^T$. The gradient of the objective at this point, in whitened coordinates, is

$$G = \frac{1}{2}S^T(I_A \oplus 0)S = \frac{1}{2} \begin{pmatrix} c^2 I_2 & csZ \\ csZ & s^2 I_2 \end{pmatrix} \oplus 0_{\text{spectators}}. \quad (6)$$

Indeed the derivative of $\sqrt{\det \Gamma_A}$ at $\Gamma_{*,A} = aI_2$ is $I_2/2$.

Proposition 4.2 (Qualified stationarity). *There is a real matrix $W \geq 0$, supported on K , such that*

$$G + W \in \mathcal{T}^\perp, \quad WD = 0, \quad W - JWJ^T = -(G - JGJ^T). \quad (7)$$

Proof. No tangent $X \in \mathcal{T}$ can simultaneously satisfy $\text{Tr}(GX) < 0$ and $P_K X P_K < 0$ on K . Otherwise $V' - e^{tX} > 0$ for sufficiently small $t > 0$: on K the difference has a strictly positive order- t term; on $K^\perp$ it starts strictly positive; and the off-diagonal Schur-complement correction is order t^2 . The same curve would lower the objective, contradicting optimality.

Separate the linear subspace

$$\{(\text{Tr}(GX), P_K X P_K) : X \in \mathcal{T}\}$$

from the open product of the negative scalar cone and the negative-definite matrix cone on K . Finite-dimensional separation gives $\alpha \geq 0$ and $W_K \geq 0$, not both zero, such that

$$\alpha G + P_K W_K P_K \in \mathcal{T}^\perp.$$

If $\alpha = 0$, pairing with X_0 from Lemma 4.1 is strictly negative for $W_K \neq 0$, a contradiction. Divide by α and write $W = P_K W_K P_K / \alpha$. If $K = \{0\}$, stationarity would instead force $G \in \mathcal{T}^\perp$, impossible because its off-diagonal $csZ/2$ block has a nonzero antisymplectic part. The support condition gives $WD = 0$, and invariance of $\mathcal{T}^\perp$ under $X \mapsto JXJ^T$ gives the last equation. $\square$

This is a constraint qualification for a nonconvex manifold problem. It is not an appeal to a convex semidefinite Slater condition.

5 Rank two and the EPR form

The support of the multiplier is more restrictive than its positivity.

Lemma 5.1. *If $P, Q \geq 0$ are real symmetric matrices with disjoint ranges, then*

$$\text{rank}(P - Q) = \text{rank } P + \text{rank } Q, \quad \text{ran}(P - Q) = \text{ran } P + \text{ran } Q.$$

Proof. If $(P - Q)x = 0$, then $Px = Qx$ lies in the intersection of the ranges, so both terms vanish. Thus $\ker(P - Q) = \ker P \cap \ker Q$. Taking orthogonal complements gives the range identity; the disjointness gives the dimension sum. The ranges need not be orthogonal. $\square$

The ranges of W and JWJ^T lie in K and JK , respectively, and are disjoint by Lemma 4.1. On the other hand, (6) shows that $G - JGJ^T$ has rank four and range exactly the real four-dimensional pair space A, D , since $cs > 0$. Equation (7) and Lemma 5.1 therefore imply

$$\text{rank } W = 2, \quad \text{ran } W + \text{ran}(JWJ^T) = \mathbb{R}_{A,D}^4. \quad (8)$$

In particular, W has no spectator support.

Write its pair blocks as

$$W_{\text{pair}} = \begin{pmatrix} A & C_0 \\ C_0^T & B \end{pmatrix}.$$

The stationarity equation requires

$$A = \alpha I_2, \quad B = \beta I_2, \quad C_0 = xI_2 + yJ_2 - \kappa Z, \quad \kappa = cs/2 > 0.$$

Here the real 2×2 basis I_2, J_2, Z, X , with $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, splits into the fixed and sign-changing parts of $C_0 \mapsto J_2 C_0 J_2^T$. Positivity forces $\alpha, \beta > 0$, since a zero diagonal block would force the corresponding off-diagonal block to vanish. Rank two makes the Schur complement zero:

$$C_0^T C_0 = \alpha \beta I_2.$$

Direct multiplication gives

$$C_0^T C_0 = (x^2 + y^2 + \kappa^2)I_2 - 2\kappa xZ - 2\kappa yX.$$

Consequently $x = y = 0$ and $\alpha\beta = \kappa^2$. There are $u, v > 0$ with $uv = \kappa$ such that

$$W_{\text{pair}} = \begin{pmatrix} u^2 I_2 & -uvZ \\ -uvZ & v^2 I_2 \end{pmatrix}. \quad (9)$$

The rank calculation has eliminated the additional number-conserving intermode terms. A generic stationary quadratic form would not suffice for the affine theorem.

6 Saturation on the retained pair

Undo only the algebraic whitening, remaining in the local Schmidt frame. Define $W_0 = S^{-T} W S^{-1}$. Complementary slackness gives

$$\text{Tr}(W_0(V - \Gamma_*)) = \text{Tr}(WD) = 0. \quad (10)$$

Factoring (9) as the Gram matrix of $(uI_2, -vZ)$ yields

$$(W_0)_{\text{pair}} = \begin{pmatrix} A_0^2 I_2 & -A_0 B_0 Z \\ -A_0 B_0 Z & B_0^2 I_2 \end{pmatrix}, \quad A_0 = uc + vs, \quad B_0 = us + vc.$$

Both coefficients are positive. Since

$$A_0 - qB_0 = u/c > 0, \quad B_0 - qA_0 = v/c > 0,$$

the gain and scale

$$t = \min\{A_0/B_0, B_0/A_0\}, \quad \eta = \max\{A_0^2, B_0^2\} \quad (11)$$

satisfy $q < t \leq 1$ and $\eta > 0$. If needed, exchange the *names* of the two retained modes when applying the two-mode inequality. Its entanglement is symmetric under that exchange. With the smaller diagonal coefficient written first,

$$(W_0)_{\text{pair}} = \eta K_t.$$

Let ρ_R be the actual two-mode marginal after the local Schmidt coordinate changes and discarding Bob's spectators. The support of W_0 , (10), and (4) imply

$$\text{Tr}(\rho_R D_t) = \text{Tr}(\rho_{\Gamma_*|A,D} D_t) = d_t(q). \quad (12)$$

This uses the actual retained marginal of the mixed state. Neither V nor its noise $V - \Gamma_*$ is required to factor across the retained and spectator modes.

The gain is in the exact admissible range of Theorem 3.1, so

$$E_F(\rho_R) \geq E(q).$$

Local Gaussian unitaries preserve pure-state entanglement. Discarding Bob modes cannot increase the measure-based convex roof, as shown explicitly in Appendix A. Therefore

$$E_F(V) \geq E_F(\rho_R) \geq E(q) = E_F^G(V) \geq E_F(V). \quad (13)$$

This proves Theorem 1.1 for strictly physical V .

As a normalization check, (11) and $uv = cs/2$ give

$$\eta = \frac{q}{2(t-q)(1-tq)}, \quad \lambda_{q,t} = -4\eta \log q.$$

For instance, when $A_0 \leq B_0$, $B_0^2(t-q)(1-tq) = (A_0 - qB_0)(B_0 - qA_0) = uv/c^2 = q/2$. The proof itself needs only saturation (12).

Corollary 6.1. *For a strictly physical one-versus-many Gaussian state, local Gaussian coordinate changes followed by discarding all but one Bob mode can preserve its entanglement of formation and Gaussian entanglement of formation.*

Proof. In the entangled case this is (13), and the retained two-mode state has equal Gaussian and unrestricted roofs. In the zero-entanglement case, local discarding preserves the value zero. $\square$

The corollary is an equality of entanglement values, not a tensor-product decomposition of the state.

7 Physical boundary covariances

Let V be any finite physical covariance, with no strictness assumption, and set $V_\epsilon = V + \epsilon I$ for $\epsilon > 0$. It is strictly physical and arises from independent local Gaussian random displacements. Refining a pure-state measure by those displacements keeps its average entanglement unchanged. Thus

$$E_F^G(V_\epsilon) = E_F(V_\epsilon) \leq E_F(V).$$

We claim that $E_F^G(V_\epsilon) \uparrow E_F^G(V)$ as $\epsilon \downarrow 0$. The minima in (1) increase monotonically and are bounded above by the value at V . For $0 < \epsilon \leq 1$, all minimizing pure covariances lie in the compact positive set bounded above by $V + I$, with a common reciprocal eigenvalue lower bound. Given any

sequence $\epsilon_j \downarrow 0$, take a convergent subsequence of its minimizers. Its limit is pure and at most V . Continuity of the one-mode entropy gives a limiting cost at least $E_F^G(V)$. Monotonicity gives the opposite bound, proving the claim.

Taking the limit in the preceding inequality yields $E_F^G(V) \leq E_F(V)$; the reverse inequality is true by restriction of the admissible decompositions. This completes the proof of Theorem 1.1. The argument uses compactness in finite covariance dimension, not an assumed continuity theorem for unrestricted entanglement of formation in infinite-dimensional Hilbert space.

8 Contribution and verification scope

The proof concerns arbitrary cross correlations in the original quadratures. It introduces no local position–momentum block form for the mixed covariance. The essential reduction is the positive rank-two multiplier, obtained from strict physicality and stationarity of the optimal pure seed.

The covariance roof, pure Gaussian normal form, and the entire affine two-mode entropy inequality are inherited ingredients. The additional one-versus-many saturation argument is not supplied by two-mode canonical forms or by symmetry-based multimode reductions. Likewise, the one-versus-many PPT/separability criteria concern the zero-entanglement case and do not determine the positive formation roof [4, Sections III–VI].

Exact rational covariance controls checked the rank identities, strict physicality, witness saturation, admissible gain and their preservation under local position–momentum shears. A pure-boundary control also confirms why the strict-feasible-tangent argument must be replaced there by Section 7. Finite examples are algebraic checks, not substitutes for the general proof or independent proofs of the imported entropy theorem.

AI assistance and verification status. This work was developed and written in an author-directed research workflow using OpenAI Codex. AI assistance included mathematical derivation, literature comparison, exact computational checks and manuscript drafting. Separate agent reviews within the same system examined the full argument and its source interfaces. Such internal review is fallible and is not external expert review or formal verification. The affine theorem of Ref. [2] remains an identified external dependency. No external acceptance is claimed.

A Local operations for arbitrary decomposition measures

We give the specific monotonicity statements used in the proof, rather than assuming that an optimal finite ensemble exists.

Let μ be any pure-state measure with barycenter a finite-energy state ρ_{AB} . A fixed local Gaussian unitary maps it to a measure with the transformed barycenter and the same component entanglements. In finitely many modes, a fixed symplectic transformation preserves finiteness of mean energy, since it bounds the transformed quadratic energy by a constant times the original energy plus a constant.

Write Bob as the retained mode D and discarded modes E . For each pure component, take the partial inner products

$$|\tilde{\psi}_{\mathbf{n}}\rangle_{AD} = (I_{AD} \otimes \langle \mathbf{n}|_E) |\psi\rangle_{ADE}$$

in a joint number basis. Their squared norms are conditional probabilities; normalize the nonzero vectors. These pure retained states form a countable decomposition of $\text{Tr}_E |\psi\rangle\langle\psi|$. Their Alice

reduced states average to ψ_A , so entropy concavity gives

$$\sum_{\mathbf{n}} \|\tilde{\psi}_{\mathbf{n}}\|^2 E\left(\frac{\tilde{\psi}_{\mathbf{n}}}{\|\tilde{\psi}_{\mathbf{n}}\|}\right) \leq S(\psi_A).$$

The amplitudes and probabilities are measurable functions of the pure state. Countable refinement of μ therefore gives an admissible retained-state measure. Tonelli applies to its nonnegative entropy cost, and the finite-energy bounds in Section 2 control the integrals. Integrating and then taking the infimum proves

$$E_F(\text{Tr}_E \rho_{ADE}) \leq E_F(\rho_{A:DE}).$$

Finally, independent local displacements act by product unitaries on each pure component and preserve its entanglement exactly. Taking the product of the component measure with a centered finite-variance Gaussian displacement measure gives the noisy barycenter and the same average cost. Choosing displacement covariance $\epsilon I/2$ changes the doubled covariance from V to $V + \epsilon I$. Infimizing proves the boundary-noise inequality used in Section 7 directly for arbitrary probability measures.

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