

Loss of Mixed Unitarity after a Positive Time in an Explicit Qutrit Semigroup

QIQC research draft

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Abstract

We give a fixed unital Lindblad generator on $\mathbb{M}_3$ whose semigroup is mixed unitary at a strictly positive time and is not mixed unitary at a later time. The two times are $s = 2\pi/\sqrt{3601}$ and $t = 3\pi/\sqrt{3601}$. At s , an exact block-evolution identity expresses the channel as a unitary conjugation composed with a semigroup generated by five Hermitian jumps. Compactness then gives membership in the finite convex hull of unitary conjugations. At t , a twisted-swap witness, valid for every complex qutrit unitary, has expectation strictly less than $-456/38125$. The construction and both certificates are analytic.

1 Question, conventions, and statement

A channel on $\mathbb{M}_d = \mathcal{L}(\mathbb{C}^d)$ is *mixed unitary* if it has a finite representation

$$\Psi(X) = \sum_{j=1}^N p_j U_j X U_j^*, \quad p_j \geq 0, \quad \sum_{j=1}^N p_j = 1, \quad (1)$$

where each U_j is unitary. We write $\text{Ad}_U(X) = UXU^*$. Throughout, $*$ denotes adjoint, T denotes transpose in the displayed computational basis, and $\bar{}$ denotes complex conjugation. Matrices in block calculations are arbitrary complex matrices; no Hermiticity restriction is imposed on their entries.

Bhat and Devendra prove eventual mixed unitarity for every dynamical semigroup of unital quantum channels [1, Theorem 4.12]. Their Theorem 5.4 describes an initial interval of nonmembership when the semigroup is not mixed unitary at every time. The subsequent open question asks whether entry into the mixed-unitary set can precede the time after which membership persists. The corresponding QIQC question [2] asks for one norm-continuous unital completely positive trace-preserving (CPTP) semigroup and two times $0 < s < t$ with membership at s and nonmembership at t .

The dissipative part used below is the qutrit construction in [1, Lemma 5.7 and Example 5.11]. The twisted-swap separation method also appears in [1, Lemmas 5.5–5.7]. Here a specified Hamiltonian produces an exactly solvable refocusing time. We include the membership and separation arguments in full.

For a map $\Psi : \mathbb{M}_3 \rightarrow \mathbb{M}_3$, use the normalized Choi convention

$$J(\Psi) = (\text{id} \otimes \Psi)(|\Omega\rangle\langle\Omega|), \quad |\Omega\rangle = \frac{1}{\sqrt{3}} \sum_{j=0}^2 |jj\rangle. \quad (2)$$

In particular, $\text{Tr } J(\Psi) = 1$ for a trace-preserving map. Let $E_{ij} = |i\rangle\langle j|$ and let $F = \sum_{i,j=0}^2 E_{ij} \otimes E_{ji}$ be the swap operator.

Theorem 1. *Set*

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \text{diag}(1, R), \quad H = \text{diag}(30, 0, 0), \quad (3)$$

and define

$$\mathcal{L}(X) = \frac{\text{Tr}(X)I_3 - BX^TB^*}{2} - X - i[H, X]. \quad (4)$$

Then $\Phi_u = e^{u\mathcal{L}}$, $u \geq 0$, is a norm-continuous unital CPTP semigroup. For

$$s = \frac{2\pi}{\sqrt{3601}}, \quad t = \frac{3\pi}{\sqrt{3601}}, \quad (5)$$

Φ_s has a finite mixed-unitary representation and Φ_t does not. More precisely, set

$$C = \text{diag}(-1, R), \quad S_C = (I_3 \otimes C)F(I_3 \otimes C^*), \quad W = S_C + \frac{I_9}{3}. \quad (6)$$

Every mixed-unitary channel Ψ satisfies $\text{Tr}(WJ(\Psi)) \geq 0$, whereas

$$\text{Tr}(WJ(\Phi_t)) = \frac{2 + e^{-3t/2} - 3e^{-t/2} - 4e^{-t/\sqrt{3601}}}{3} < -\frac{456}{38125}. \quad (7)$$

2 A unital Lindblad generator

Define the map

$$T(X) = \frac{\text{Tr}(X)I_3 - BX^TB^*}{2}. \quad (8)$$

Its complete positivity can be checked directly from

$$J(T) = \frac{I_9 - (I_3 \otimes B)F(I_3 \otimes B^*)}{6} \geq 0. \quad (9)$$

Indeed, B is unitary, and the conjugated swap is a Hermitian involution. Direct evaluation gives $T(I_3) = I_3$ and $\text{Tr} T(X) = \text{Tr} X$.

For an explicit Lindblad representation, put

$$L_{ij} = \frac{B(E_{ij} - E_{ji})}{\sqrt{2}}, \quad 0 \leq i < j \leq 2. \quad (10)$$

Expanding the three terms yields

$$T(X) = \sum_{i < j} L_{ij} X L_{ij}^*, \quad \sum_{i < j} L_{ij}^* L_{ij} = I_3, \quad \sum_{i < j} L_{ij} L_{ij}^* = I_3. \quad (11)$$

For example, before multiplication by B , the first identity is $\sum_{i < j} (E_{ij} - E_{ji})X(E_{ij} - E_{ji})^* = \text{Tr}(X)I_3 - X^T$. Consequently

$$\mathcal{L}(X) = -i[H, X] + \sum_{i < j} \left(L_{ij} X L_{ij}^* - \frac{1}{2} \{L_{ij}^* L_{ij}, X\} \right), \quad (12)$$

where $\{A, X\} = AX + XA$.

The semigroup properties also follow without invoking a generator classification. The Poisson expansion

$$e^{u(T-\text{id})} = e^{-u} \sum_{n=0}^{\infty} \frac{u^n}{n!} T^n \quad (13)$$

is a norm limit of unital CPTP maps, as is seen by assigning any truncated tail weight to id . The Hamiltonian part generates $\text{Ad}_{e^{-iuH}}$. The finite-dimensional Lie product formula gives

$$e^{u\mathcal{L}} = \lim_{n \rightarrow \infty} \left(e^{(u/n)(T-\text{id})} \circ \text{Ad}_{e^{-i(u/n)H}} \right)^n. \quad (14)$$

Complete positivity, trace preservation and unitality survive this norm limit. Finally, finite-dimensional matrix exponentiation gives $\Phi_0 = \text{id}$, $\Phi_{u+v} = \Phi_u \circ \Phi_v$, and norm continuity.

3 Exact block evolution and refocusing

Write

$$X = \begin{pmatrix} x & r \\ c & A \end{pmatrix}, \quad r \in \mathbb{C}^{1 \times 2}, \quad c \in \mathbb{C}^{2 \times 1}, \quad A \in \mathbb{M}_2.$$

The elementary identity

$$RA^T R^* = \text{Tr}(A)I_2 - A \quad (15)$$

holds for every complex 2×2 matrix. Substitution into (4) gives all four block equations:

$$x' = \frac{1}{2} \text{Tr}(A) - x, \quad A' = \frac{x}{2} I_2 - \frac{1}{2} A, \quad (16)$$

$$r' = (-1 - 30i)r + \frac{1}{2} c^T R, \quad c' = (-1 + 30i)c - \frac{1}{2} R r^T. \quad (17)$$

Here is the complete solution of the diagonal blocks. Write the initial data as x_0, r_0, c_0, A_0 , let $\tau = x_0 + \text{Tr } A_0$, and put

$$v(u) = e^{-3u/2}, \quad w(u) = e^{-u/2}. \quad (18)$$

Since $x + \text{Tr } A = \tau$ is constant, the solution of (16) is

$$x(u) = \frac{\tau}{3} + \left(x_0 - \frac{\tau}{3}\right) v(u), \quad (19)$$

$$A(u) = w(u) \left(A_0 - \frac{\text{Tr } A_0}{2} I_2\right) + \frac{\tau - x(u)}{2} I_2. \quad (20)$$

To solve the other blocks, set $q = c^T R$. Equation (17) implies $q' = (-1 + 30i)q - r/2$, so

$$\frac{d}{du} \begin{pmatrix} r \\ q \end{pmatrix} = (-I_2 + M) \begin{pmatrix} r \\ q \end{pmatrix}, \quad M = \begin{pmatrix} -30i & 1/2 \\ -1/2 & 30i \end{pmatrix}. \quad (21)$$

The displayed scalar matrix acts componentwise on the two rows. With $\omega = \sqrt{3601}/2$, one has $M^2 = -\omega^2 I_2$, hence

$$e^{u(-I_2+M)} = e^{-u} \left(\cos(\omega u) I_2 + \frac{\sin(\omega u)}{\omega} M \right). \quad (22)$$

Equivalently, define

$$a(u) = \cos(\omega u) - i \frac{30}{\omega} \sin(\omega u), \quad k(u) = \frac{\sin(\omega u)}{2\omega}. \quad (23)$$

The remaining blocks are

$$r(u) = e^{-u} (a(u)r_0 + k(u)c_0^T R), \quad (24)$$

$$c(u) = e^{-u} (\overline{a(u)}c_0 - k(u)Rr_0^T). \quad (25)$$

The conjugate of $a(u)$ is a scalar coefficient; these formulas impose no relation between r_0 and c_0 .

Consider the auxiliary generator

$$\mathcal{D}_0 \begin{pmatrix} x & r \\ c & A \end{pmatrix} = \begin{pmatrix} \text{Tr}(A)/2 - x & -r \\ -c & xI_2/2 - A/2 \end{pmatrix}. \quad (26)$$

Its diagonal equations agree with (16), while both off-diagonal blocks are multiplied by e^{-u} under $e^{u\mathcal{D}_0}$. At $s = \pi/\omega$, (22) is exactly $-e^{-s} I_2$. Thus, for $Q = \text{diag}(-1, 1, 1)$,

$$\boxed{\Phi_s = \text{Ad}_Q \circ e^{s\mathcal{D}_0}.} \quad (27)$$

This equality holds on all of $\mathbb{M}_3$. It compares two channels at one specified time and does not assume that their generators coincide.

4 Finite mixed-unitary membership at the positive time

For any matrix V , use the dissipator notation

$$\mathcal{D}_V(X) = VXV^* - \frac{1}{2}(V^*VX + XV^*V). \quad (28)$$

Let $P = \text{diag}(0, 1, 1)$. Direct block multiplication gives

$$\mathcal{D}_0 = \sum_{a=1}^2 \left(\mathcal{D}_{E_{0a}/\sqrt{2}} + \mathcal{D}_{E_{a0}/\sqrt{2}} \right) + \mathcal{D}_{P/\sqrt{2}}. \quad (29)$$

The paired terms give the diagonal part in (26) and off-diagonal decay $-3r/4, -3c/4$. The last term contributes the remaining $-r/4, -c/4$ and leaves the diagonal blocks unchanged. Define the Hermitian matrices

$$A_a = \frac{E_{0a} + E_{a0}}{2}, \quad K_a = \frac{i(E_{0a} - E_{a0})}{2}, \quad a = 1, 2. \quad (30)$$

For each a , expansion of the two dissipators shows that

$$\mathcal{D}_{E_{0a}/\sqrt{2}} + \mathcal{D}_{E_{a0}/\sqrt{2}} = \mathcal{D}_{A_a} + \mathcal{D}_{K_a}. \quad (31)$$

Thus $\mathcal{D}_0$ is a sum of five Hermitian-jump generators.

Lemma 1. *If $D = \sum_{j=1}^m \mathcal{D}_{V_j}$ with $V_j = V_j^*$ in a fixed finite dimension, then e^{uD} is mixed unitary in the finite-mixture sense for every $u \geq 0$.*

Proof. For one Hermitian V , $\mathcal{D}_V(X) = -[V, [V, X]]/2$. In an eigenbasis of V , with eigenvalues λ_a , its exponential multiplies the (a, b) matrix entry by $\exp[-u(\lambda_a - \lambda_b)^2/2]$. For $u > 0$ this is the characteristic function of a centered Gaussian of variance u , giving

$$e^{u\mathcal{D}_V}(X) = \frac{1}{\sqrt{2\pi u}} \int_{\mathbb{R}} e^{-z^2/(2u)} e^{-izV} X e^{izV} dz. \quad (32)$$

At $u = 0$ the map is the identity.

The set of unitary conjugations is compact in the finite-dimensional real vector space of Hermiticity-preserving maps. Its convex hull is also compact: by Caratheodory's theorem in real dimension N , every point in the hull uses at most $N + 1$ points, so the hull is a continuous image of a compact set. Approximating the Gaussian integral by finite convex sums and using this closedness shows that $e^{u\mathcal{D}_V}$ lies in that convex hull. In particular, the exact channel has a finite representation of the form (1).

Compositions of finite mixed-unitary maps remain mixed unitary, since $\text{Ad}_U \circ \text{Ad}_V = \text{Ad}_{UV}$. The Lie product formula now gives

$$e^{uD} = \lim_{n \rightarrow \infty} \left(e^{(u/n)\mathcal{D}_{V_1}} \circ \dots \circ e^{(u/n)\mathcal{D}_{V_m}} \right)^n. \quad (33)$$

Every approximant is mixed unitary, and closedness gives the same conclusion for the limit, with a finite convex representation. $\square$

Equations (29)–(31) and Lemma 1 show that $e^{s\mathcal{D}_0}$ is mixed unitary. The exact identity (27) therefore proves mixed unitarity of Φ_s . The finite decomposition is established by existence in the finite-dimensional convex hull; no infinite-mixture enlargement of definition (1) is used.

5 A witness for all complex unitary channels

The Hermitian matrix W in (6) separates the later channel from every mixed-unitary channel. We first establish its inequality globally.

Lemma 2. *For every $U \in \mathbb{U}(3)$, $\text{Tr}(WJ(\text{Ad}_U)) \geq 0$. Consequently the same inequality holds for every mixed-unitary channel.*

Proof. Put $V = C^*U$, which is an arbitrary complex unitary. From (2) and the action of the swap,

$$\text{Tr}(S_C J(\text{Ad}_U)) = \langle \Omega | (I_3 \otimes V^*) F(I_3 \otimes V) | \Omega \rangle = \frac{1}{3} \text{Tr}(V^* V^T). \quad (34)$$

Let $V_+ = (V + V^T)/2$ and $V_- = (V - V^T)/2$. The symmetric and skew-symmetric parts are orthogonal for the complex Frobenius inner product: terms with interchanged indices cancel. Therefore

$$\text{Tr}(V^* V^T) = \|V_+\|_F^2 - \|V_-\|_F^2 = 3 - 2\|V_-\|_F^2. \quad (35)$$

In odd dimension three, $(V_-)^T = -V_-$ implies $\det V_- = -\det V_-$, hence $\text{rank } V_- \leq 2$. Both V and V^T are unitary, so

$$\|V_-\|_{\text{op}} \leq \frac{\|V\|_{\text{op}} + \|V^T\|_{\text{op}}}{2} = 1, \quad \|V_-\|_F^2 \leq 2.$$

Equations (34)–(35) give a lower bound $-1/3$ for the swap expectation. Since $\text{Tr } J(\text{Ad}_U) = 1$, the $I_9/3$ term in W shifts this bound to zero. Linearity proves the result for every finite convex combination (1). $\square$

6 Exact later-time evaluation and a rational negative bound

We evaluate the witness from the full block solution, with no restriction on the tested unitary family. For brevity write $v = v(u)$, $w = w(u)$ and $k = k(u)$. Equations (19)–(20) give

$$[\Phi_u(E_{00})]_{00} = \frac{1+2v}{3}, \quad [\Phi_u(E_{bd})]_{\text{lower}} = wE_{bd}^{(2)} + \delta_{bd}\gamma I_2, \quad b, d \in \{1, 2\}, \quad (36)$$

where $\gamma = (2+v)/6 - w/2$, and $E_{bd}^{(2)}$ is the matrix unit in the lower 2×2 block.

Expanding the normalized Choi matrix and the swap yields the four-index contraction

$$\text{Tr}(S_C J(\Phi_u)) = \frac{1}{3} \sum_{i,j,a,b=0}^2 C_{aj} \overline{C_{bi}} [\Phi_u(E_{ji})]_{ba}. \quad (37)$$

To spell out this calculation, split the sum *inside* its factor $1/3$ into four groups according to whether i, j are zero or in $\{1, 2\}$. They contribute

$$i = j = 0 : \quad \frac{1+2v}{3}, \quad (38)$$

$$i, j \in \{1, 2\} : \quad -2w + 2\gamma = \frac{2+v}{3} - 3w, \quad (39)$$

$$i = 0, j \in \{1, 2\} : \quad 2e^{-u}k, \quad (40)$$

$$i \in \{1, 2\}, j = 0 : \quad 2e^{-u}k. \quad (41)$$

For (39), use $\sum_{i,j=1}^2 R_{ij}R_{ji} = -2$ and $\sum_{a,i=1}^2 R_{ai}^2 = 2$ in (36). For (40), the input E_{j0} has $c_0 = e_j$ and $r_0 = 0$, so (24) gives $[\Phi_u(E_{j0})]_{0a} = e^{-u}kR_{ja}$; the factor $C_{00} = -1$ then gives $-e^{-u}k \text{Tr}(R^2) = 2e^{-u}k$. For (41), equation (25) gives $[\Phi_u(E_{0i})]_{b0} = -e^{-u}kR_{bi}$, and the sum is $e^{-u}k \text{Tr}(R^*R) = 2e^{-u}k$.

Adding the four groups and the identity term in W , with $\text{Tr } J(\Phi_u) = 1$, proves the exact formula

$$\text{Tr}(WJ(\Phi_u)) = \frac{2 + e^{-3u/2} - 3e^{-u/2} + 4e^{-u}k(u)}{3}. \quad (42)$$

At the later time $t = 3\pi/(2\omega)$, one has $\sin(\omega t) = -1$ and $k(t) = -1/\sqrt{3601}$. This gives the equality in (7).

It remains to prove the strict sign using rational bounds. Put $y = e^{-t/2} \in (0, 1)$. Then

$$2 + e^{-3t/2} - 3e^{-t/2} = 2 + y^3 - 3y = (1 - y)^2(y + 2) \leq \frac{3t^2}{4}, \quad (43)$$

using $1 - e^{-t/2} \leq t/2$ and $y + 2 \leq 3$. The elementary estimates $60 < \sqrt{3601} < 61$ and $\pi < 22/7$ imply

$$0 < t < \frac{11}{70} < \frac{4}{25}. \quad (44)$$

Consequently the positive expression in (43) is strictly smaller than $12/625$. On the other hand,

$$\frac{4e^{-t}}{\sqrt{3601}} > \frac{4(1-t)}{61} > \frac{84}{1525}, \quad (45)$$

where $e^{-t} > 1 - t$ and (44) were used. Combining these inequalities gives

$$\text{Tr}(WJ(\Phi_t)) < \frac{1}{3} \left(\frac{12}{625} - \frac{84}{1525} \right) = -\frac{456}{38125} < 0. \quad (46)$$

Lemma 2 now rules out every mixed-unitary representation of Φ_t . Together with (27), Lemma 1, and $0 < s < t$, this completes the proof of Theorem 1.

7 Scope of the construction

The same fixed generator is used at both times. Membership at s follows from an exact equality of linear maps and compactness of the finite convex hull. Nonmembership at t follows from a witness valid on all complex unitary channels and a strict rational inequality. Floating-point reproductions serve only as supplementary error checks and are not premises of this proof.

The result addresses positive-time loss of mixed unitarity. It does not identify the first positive mixed-unitary time or the eventual membership threshold, and it does not characterize all intermediate membership times. In particular, no discreteness assertion about that set is made. The eventual-mixed-unitarity theorem of Bhat and Devendra is consistent with the later failure exhibited here. Priority and broader literature novelty remain separate from the stated analytic claim.

References

- [1] B. V. Rajarama Bhat and R. Devendra, *On regions of mixed unitarity for semigroups of unital quantum channels*, arXiv:2512.23598v3, revised 9 September 2026. Primary full text; doi:10.48550/arXiv.2512.23598.
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