Optimal precision dependence of low-energy Hamiltonian simulation

Yuxuan Zhang

Institute of Physics, École Polytechnique Fédérale de Lausanne (EPFL),
CH-1015 Lausanne, Switzerland

Department of Physics, Princeton University,
Princeton, New Jersey 08544, USA

21 September 2026

Research draft prepared with AI assistance.
Independent expert review and publication priority remain unconfirmed.

Abstract

We study the query complexity of simulating $H = \lambda A^\dagger A$ on states of energy at most Δ, given an exact block encoding of the contraction A and controlled access to that encoding and its inverse. Set $T = t\lambda$, $s = t\Delta$, and $L = \log(1/\epsilon)$. For every asymptotic family with $\epsilon \to 0$, $\epsilon = o(s)$, $s = o(L)$, and $L = o(T)$, we establish the worst-case query complexity

$$\Theta\left(\sqrt{\frac{TL}{\log(2 + L/s)}}\right)$$

for phase-sensitive vector error at most ϵ, including the requirement that all workspace be returned to its initial state up to that error. The upper bound combines a powered reflection walk with an explicit polynomial that is accurate on the promised spectral interval and contractive on the full interval. A Laurent-polynomial unitary completion controls ancillary leakage. For the lower bound, averaging over four scalar reflection encodings reduces an arbitrary query circuit to a bounded complex polynomial; finite differences and a derivative estimate give a matching degree bound. In particular, fixed $s > 0$ gives $\Theta(\sqrt{TL/\log L})$ queries. The result concerns exact factor access and unrestricted oracle-independent gates; it does not establish a corresponding gate-complexity bound.

1 Introduction and main result

The low-energy promise can change the cost of Hamiltonian simulation when the oracle exposes a factor of the Hamiltonian. For $H = \lambda A^\dagger A$, Zlokapa and Somma give the query upper bound

$$O\left(t\sqrt{\lambda\Gamma} + \sqrt{\frac{\lambda}{\Gamma}}\log\frac{1}{\epsilon}\right), \qquad \Delta \le \Gamma \le \lambda. \tag{1.1}$$

Their polynomial construction, with tolerance of order ϵ^2, also gives the vector-error convention used below without changing this asymptotic expression [1, Lemma 1.2 and Sec. 2.2]. In the intermediate regime $t\Delta \ll \log(1/\epsilon) \ll t\lambda$, choosing $\Gamma = \log(1/\epsilon)/t$ gives $O(\sqrt{t\lambda \log(1/\epsilon)})$ queries. The precision dependence is left open in Ref. [1, Secs. 1 and 7].

This paper determines the precision dependence in the exact factor-oracle model specified below. The improvement comes from matching a polynomial to a small spectral interval while retaining contractivity everywhere the oracle may have spectrum. A binomial cutoff enforces that global constraint without sacrificing the approximation order at zero. The matching lower bound uses the same separation between local accuracy and global boundedness. The unitary synthesis step is a standard quantum signal processing principle [2]; we include a proof to make the query count and clean-output guarantee explicit.

1.1 Oracle model and error convention

Let $A \in \mathbb{C}^{N\times N}$ satisfy $\|A\| \le 1$, and let

$$H = \lambda A^\dagger A, \qquad P_\Delta = \mathbf{1}_{[0,\Delta]}(H), \qquad \lambda > 0, \quad 0 < \Delta \le \lambda. \tag{1.2}$$

The oracle is a unitary V_A with m encoding ancillas such that

$$(\langle 0^m| \otimes I_N) V_A (|0^m\rangle \otimes I_N) = A. \tag{1.3}$$

Each use of V_A, $V_A^\dagger$, or a controlled version costs one query. Other gates are unrestricted but must be independent of the unknown oracle. They may depend on $t, \lambda, \Delta, \epsilon, N$, and m. State preparation is excluded from the query count. No classical description of A and no free implementation of P_Δ are supplied.

For $t > 0$ and $0 < \epsilon < 1/2$, a simulator is a unitary W with workspace initialized to $|0^a\rangle$, required to obey

$$\sup_{\substack{\|\psi\|=1 \\ P_\Delta|\psi\rangle = |\psi\rangle}} \left\| W(|0^a\rangle|\psi\rangle) - |0^a\rangle \mathrm{e}^{-\mathrm{i}tH}|\psi\rangle \right\| \le \epsilon. \tag{1.4}$$

This condition is phase sensitive. It also includes all amplitude that leaks into nonzero workspace states. The circuit must work for every admissible factor and every exact encoding of that factor. Let $Q_{\mathrm{opt}}(T, s, \epsilon)$ denote the resulting worst-case query complexity, with no restriction on the system or encoding dimensions. This is the model of QIQCOP problem `op_25d9dee5435ea835` [3].

All logarithms are natural. We use

$$T = t\lambda, \qquad s = t\Delta, \qquad L = \log(1/\epsilon), \qquad D = \log(2 + L/s). \tag{1.5}$$

The condition $\epsilon = o(s)$ excludes a regime in which the identity already achieves the requested error: indeed, $\left\|(\mathrm{e}^{-\mathrm{i}tH} - I)P_\Delta\right\| \le t\Delta = s$.

Theorem 1.1 (Optimal precision dependence)**.** *Along every asymptotic family satisfying*

$$\epsilon \longrightarrow 0, \qquad \epsilon = o(s), \qquad s = o(L), \qquad L = o(T), \tag{1.6}$$

the query complexity in the above model is

$$Q_{\mathrm{opt}}(T, s, \epsilon) = \Theta\left(\sqrt{\frac{TL}{D}}\right) = \Theta\left(\sqrt{\frac{t\lambda \log(1/\epsilon)}{\log(2 + \log(1/\epsilon)/(t\Delta))}}\right). \tag{1.7}$$

The implied constants are absolute and independent of N and m. The upper and lower inequalities hold eventually along each such family; the point at which they apply may depend on the family.

Since $D \to \infty$, the scale in Eq. (1.7) is smaller than $\sqrt{TL}$ by a factor $\sqrt{D}$ throughout Eq. (1.6). Table 1 records three consequences. The last row is useful for the proof: L/D need not diverge, so rounding the polynomial degree and retaining the first finite difference are necessary.

2 A locally accurate, globally contractive polynomial

The upper bound rests on the following explicit construction. Local Taylor approximation alone is insufficient: a polynomial used as a block of a unitary must remain bounded on the full spectral interval. The cutoff below supplies that bound.

Table 1: Consequences of Theorem 1.1. In all rows, $\epsilon = \mathrm{e}^{-L}$, $L \to \infty$, and $L = o(T)$.

Energy-time parameter s	Denominator D	Optimal queries
Fixed $s > 0$	$D \sim \log L$	$\Theta(\sqrt{TL/\log L})$
$s = L/\log L$	$D \sim \log\log L$	$\Theta(\sqrt{TL/\log\log L})$
$s = \epsilon L$	$D \sim L$	$\Theta(\sqrt{T})$

Lemma 2.1 (Contractive approximation)**.** *Let $M, r \geq 1$ be integers, and suppose $T > 0$ satisfies $T/r^2 \leq M/32$. Define*

$$h(z) = T \sin^2\left(\frac{\arcsin\sqrt{z}}{r}\right), \qquad 0 \leq z \leq 1. \tag{2.1}$$

There is an explicit polynomial P of degree at most $17M - 1$ such that

$$|P(z)| \leq 1, \qquad \left|P(z) - \mathrm{e}^{-\mathrm{i}h(z)}\right| \leq (64z)^M \quad (0 \leq z \leq 1). \tag{2.2}$$

Proof. First expand $h(z) = \sum_{j\geq 1} b_j z^j$ at zero. The coefficients obey

$$b_1 = \frac{T}{r^2}, \qquad b_{j+1} = \frac{j^2 - r^{-2}}{(j+1)(j+1/2)} b_j \quad (j \geq 1). \tag{2.3}$$

For completeness, the function $y(z) = \cos(2\alpha \arcsin\sqrt{z})$ solves

$$z(1-z)y'' + (1/2 - z)y' + \alpha^2 y = 0.$$

Substituting $\alpha = 1/r$ and $h = T(1-y)/2$ gives Eq. (2.3). Every b_j is nonnegative. Taking $z \uparrow 1$ and using monotone convergence yields

$$\sum_{j\geq 1} b_j = T\sin^2\left(\frac{\pi}{2r}\right) \leq \frac{\pi^2 T}{4r^2} \leq \frac{\pi^2 M}{128} < \frac{M}{8}. \tag{2.4}$$

Write $\mathrm{e}^{-\mathrm{i}h(z)} = \sum_{j\geq 0} c_j z^j$ and truncate at degree $M-1$:

$$p(z) = \sum_{j=0}^{M-1} c_j z^j, \qquad E = \mathrm{e}^{M/8}. \tag{2.5}$$

The exponential power series and Eq. (2.4) give $\sum_{j\geq 0} |c_j| \leq E$. Absolute convergence also holds at $z = 1$. Thus, throughout $[0,1]$,

$$|p(z)| \leq E, \qquad |p(z) - \mathrm{e}^{-\mathrm{i}h(z)}| \leq E z^M. \tag{2.6}$$

The coefficients are computable by $c_0 = 1$ and $c_n = -\mathrm{i}\sum_{j=1}^{n} j b_j c_{n-j}/n$.

Now set

$$f(z) = \sum_{j=0}^{M-1} \binom{16M}{j} z^j (1-z)^{16M-j}, \qquad P(z) = f(z)p(z). \tag{2.7}$$

For real $z \in [0,1]$, $f(z)$ is the probability that $X \sim \mathrm{Bin}(16M, z)$ satisfies $X \leq M - 1$. If $z \leq 1/8$, the single term $X = M$ gives

$$\begin{aligned} 1 - f(z) &\geq \binom{16M}{M} z^M (1-z)^{15M} \\ &\geq \left[16(7/8)^{15}\right]^M z^M \geq 2^M z^M \geq E z^M. \end{aligned} \tag{2.8}$$

Here $\binom{16M}{M} \geq 16^M$, and $16(7/8)^{15} > 2$. Since $1 - f(z) \leq 1$, the quantity $1 - Ez^M$ is nonnegative on this interval. Equation (2.6) therefore gives

$$|P(z)| \leq (1 - Ez^M)(1 + Ez^M) \leq 1.$$

If instead $z \geq 1/8$, Markov's inequality applied to 2^{-X} gives

$$f(z) \leq 2^M(1 - z/2)^{16M} \leq \mathrm{e}^{(\log 2 - 1)M} \leq \mathrm{e}^{-M/4}. \tag{2.9}$$

Together with $|p(z)| \leq E$, this proves $|P(z)| \leq \mathrm{e}^{-M/8} \leq 1$.

Finally, the factorial-moment bound gives

$$1 - f(z) = \Pr(X \geq M) \leq \mathbb{E}\binom{X}{M} = \binom{16M}{M} z^M \leq (16\mathrm{e}z)^M.$$

Using $0 \leq f \leq 1$ and $|\mathrm{e}^{-\mathrm{i}h(z)}| = 1$,

$$\begin{aligned} |P(z) - \mathrm{e}^{-\mathrm{i}h(z)}| &\leq f(z)|p(z) - \mathrm{e}^{-\mathrm{i}h(z)}| + 1 - f(z) \\ &\leq Ez^M + (16\mathrm{e}z)^M \leq (64z)^M. \end{aligned}$$

The degree of P is at most $16M + (M - 1)$, as claimed. □

3 Unitary realization and the factor walk

3.1 Exact Laurent-polynomial synthesis

The following synthesis fact is standard in quantum signal processing; see, for example, the product decomposition in Ref. [2]. We use a degree bound with nonoptimized constants and give the construction directly.

Lemma 3.1 (Unitary completion). *Let P have degree at most d and satisfy $|P(z)| \leq 1$ on $[0, 1]$. For any unitary B, put $Z = (2I - B - B^\dagger)/4$. There is a unitary W on one additional qubit and the space of B with*

$$(\langle 0| \otimes I)W(|0\rangle \otimes I) = P(Z). \tag{3.1}$$

It uses at most $6d$ calls to B or $B^\dagger$, including controlled calls, and otherwise only gates independent of B.

Proof. Set

$$a(w) = P\left(\frac{2 - w - w^{-1}}{4}\right).$$

This Laurent polynomial is supported in $[-d, d]$ and has modulus at most one on the unit circle. The nonnegative trigonometric polynomial $1 - |a(w)|^2$ has degree at most $2d$. By scalar Fejér–Riesz factorization, it equals $|b(w)|^2$ there for an ordinary polynomial b of degree at most $2d$. One way to see the factorization is to pair roots under reciprocal conjugation and choose one from each pair; roots on the unit circle have even multiplicity. A nonnegative constant is absorbed into the factor. If the residual vanishes identically, take $b = 0$.

For a Laurent polynomial g, define $g^*(w) = \overline{g(1/\overline{w})}$. Then

$$U(w) = \begin{pmatrix} a(w) & -b^*(w) \\ b(w) & a^*(w) \end{pmatrix} \tag{3.2}$$

is unitary on the unit circle and is supported in $[-2d, 2d]$. Consequently $C(w) = w^{2d}U(w)$ is a polynomial matrix of degree at most $4d$, also unitary on the circle.

We now factor an arbitrary polynomial matrix $C(w) = \sum_{j=0}^{n} C_j w^j$ that is unitary on the circle. For $n > 0$, let E be the orthogonal projection onto $\operatorname{ran} C_n$. The coefficient identity $C_0^\dagger C_n = 0$ implies $EC_0 = 0$. Hence

$$(I - E + w^{-1}E)C(w)$$

has no negative powers and has degree at most $n - 1$, since $(I - E)C_n = 0$. It remains unitary on the circle. Induction factors C into at most n matrices $I - E + wE$ and a constant unitary.

Substituting B for w implements each factor with one projection-controlled call to B on the synthesis qubit. Rank-one projections require a constant change of basis, while rank-two projections require an unconditional call. At most $4d$ such calls implement $C(B)$. Multiplying by B^{-2d} uses $2d$ further inverse calls and gives $U(B)$ exactly. Its top-left block is $a(B) = P(Z)$, proving Eq. (3.1). □

3.2 Recovering the low-energy spectrum

Let $J|u\rangle = |0^m\rangle|u\rangle$ embed the system into the encoding space, and define

$$\Pi = JJ^\dagger, \quad R = 2\Pi - I, \quad \Pi' = V_A^\dagger \Pi V_A, \quad R' = 2\Pi' - I, \quad B = (-R'R)^r. \tag{3.3}$$

The reflection R is oracle independent, and $R' = V_A^\dagger R V_A$. Thus B or $B^\dagger$ costs $2r$ factor queries. Its controlled version has the same query cost under the stated access convention.

Lemma 3.2 (Walk action)**.** *Suppose $A^\dagger A|u\rangle = \sigma^2|u\rangle$ with $\|u\| = 1$ and $0 \le \sigma < 1$. On the invariant subspace generated by $J|u\rangle$ and $\Pi' J|u\rangle$, the operator $Z = (2I - B - B^\dagger)/4$ is scalar:*

$$Z = \sin^2(r \arcsin \sigma) I. \tag{3.4}$$

Such subspaces associated with orthogonal eigenvectors are orthogonal.

Proof. For $0 < \sigma < 1$, put $v = J|u\rangle$ and

$$w = \frac{\Pi' v - \sigma^2 v}{\sigma\sqrt{1 - \sigma^2}}.$$

The identity $J^\dagger \Pi' J = A^\dagger A$ shows that v, w are orthonormal. In this basis,

$$R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad R' = \begin{pmatrix} 2\sigma^2 - 1 & 2\sigma\sqrt{1 - \sigma^2} \\ 2\sigma\sqrt{1 - \sigma^2} & 1 - 2\sigma^2 \end{pmatrix}.$$

The eigenvalues of $-R'R$ are $\mathrm{e}^{\pm 2\mathrm{i} \arcsin \sigma}$. Taking the rth power and its adjoint gives Eq. (3.4) on the whole plane. If $\sigma = 0$, then $\Pi' v = 0$, so $Bv = v$ and $Zv = 0$ directly. For orthonormal eigenvectors, the identity $\langle v_j, \Pi' v_\ell \rangle = \sigma_\ell^2 \delta_{j\ell}$, together with $\Pi'^2 = \Pi'$, proves orthogonality of the generated subspaces. □

4 Upper bound

We first give a finite-parameter statement. It separates the approximation and leakage estimates from the asymptotic choice of M and r.

Proposition 4.1 (Finite-parameter simulator)**.** *Let $0 < s < T$, and choose integers $M, r \geq 1$ such that*

$$\frac{T}{r^2} \leq \frac{M}{32}, \qquad \theta_* := r \arcsin \sqrt{s/T} \leq \frac{\pi}{2}. \tag{4.1}$$

There is a simulator using at most $12(17M-1)r$ factor queries whose error in Eq. (1.4) is at most

$$\sqrt{2\delta}, \qquad \delta = \left(64 \sin^2 \theta_*\right)^M. \tag{4.2}$$

Proof. Use the polynomial of Lemma 2.1, the walk in Eq. (3.3), and the unitary in Lemma 3.1. For a promised eigenvector, $\sigma^2 \leq s/T$ and $0 \leq r \arcsin \sigma \leq \pi/2$. Therefore

$$z = \sin^2(r \arcsin \sigma), \qquad h(z) = T\sigma^2.$$

The inverse sine uses its principal branch, and the angle condition ensures the displayed identity is exact. Lemmas 2.1 and 3.2, including orthogonality of the invariant subspaces, give uniformly for promised unit vectors

$$\left\| P(Z)J|\psi\rangle - J\mathrm{e}^{-\mathrm{i}tH}|\psi\rangle \right\| \leq \delta. \tag{4.3}$$

Let $\eta = W(|0\rangle J|\psi\rangle)$ denote the full output, and let $\eta_* = |0\rangle J\mathrm{e}^{-\mathrm{i}tH}|\psi\rangle$ denote the ideal clean output. Both are unit vectors. By the top-left block identity and Eq. (4.3),

$$\mathrm{Re}\langle \eta_*, \eta\rangle \geq 1 - \delta, \qquad \|\eta - \eta_*\|^2 = 2 - 2\,\mathrm{Re}\langle \eta_*, \eta\rangle \leq 2\delta.$$

This controls all workspace leakage; no measurement, postselection, or implementation of P_Δ is used. The query count follows from $6d$ walk calls, $d \leq 17M - 1$, and $2r$ factor queries per call. □

Proof of the upper bound in Theorem 1.1. Choose

$$M = \left\lceil \frac{6L}{D} \right\rceil, \qquad r = \left\lceil \sqrt{\frac{32T}{M}} \right\rceil. \tag{4.4}$$

We retain the ceiling operations throughout, since M can remain bounded along an admissible family.

First, $s = o(L)$ gives $D \to \infty$. Since $\epsilon = o(s)$, eventually $s \geq \epsilon = \mathrm{e}^{-L}$, whence

$$D \leq \log(2 + L\mathrm{e}^L) = L + \log L + o(1). \tag{4.5}$$

It follows that L/D is bounded below by a positive absolute constant and

$$M = \Theta(L/D), \qquad M = o(T), \qquad M/s \longrightarrow \infty. \tag{4.6}$$

For the last limit, write $x = L/s \to \infty$ and use $M/s \geq 6x/\log(2+x)$.

Eventually $r \leq 2\sqrt{32T/M}$. Also $s/T \to 0$, so $\arcsin \sqrt{s/T} \leq 2\sqrt{s/T}$ eventually. Consequently

$$\theta_*^2 \leq \frac{512s}{M} = o(1). \tag{4.7}$$

The branch condition in Proposition 4.1 is satisfied, and its polynomial error is at most

$$\delta \leq \left(\frac{32768s}{M}\right)^M. \tag{4.8}$$

To check the precision uniformly, observe that MD/L remains between two positive absolute constants. Thus

$$\frac{\log(M/(32768s))}{D} = \frac{\log(L/s) - \log D + \log(MD/L) - \log 32768}{D} \longrightarrow 1. \tag{4.9}$$

Eventually the numerator is at least $D/2$, so $M\log(M/(32768s)) \geq 3L$. Equation (4.8) gives $\delta \leq \mathrm{e}^{-3L} = \epsilon^3$. Since $\epsilon < 1/2$, $\sqrt{2\delta} \leq \epsilon$, exactly as required by Eq. (1.4).

Finally,

$$12(17M-1)r = O(\sqrt{TM}) = O\left(\sqrt{\frac{TL}{D}}\right).$$

All constants used here are independent of N, m, and of the particular asymptotic family. $\square$

5 Lower bound

5.1 A bounded polynomial from arbitrary query circuits

Lemma 5.1 (Four-sign reduction)**.** *Suppose a Q-query circuit satisfies Eq. (1.4) for all admissible oracles. There is a complex polynomial p of degree at most $\lfloor Q/2 \rfloor$ satisfying*

$$|p(y)| \leq 1 \quad (0 \leq y \leq 1), \qquad |p(y) - \mathrm{e}^{-\mathrm{i}Ty}| \leq \epsilon \quad (0 \leq y \leq s/T). \tag{5.1}$$

Proof. Consider scalar factors $A = a$ and their one-ancilla reflection encodings

$$V(a,b) = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \qquad a, b \in \mathbb{R}, \quad a^2 + b^2 = 1. \tag{5.2}$$

The return amplitude $F(a,b)$ of the circuit on its fully initialized input is a polynomial of total degree at most Q. Each query has entries of degree at most one in a, b, including controlled and inverse queries, while the intervening gates are independent of a, b. Arbitrary workspace does not change the degree count.

Average over independent signs:

$$p(y) = \frac{1}{4} \sum_{\alpha,\beta \in \{-1,1\}} F(\alpha\sqrt{y}, \beta\sqrt{1-y}). \tag{5.3}$$

Odd powers of either variable vanish. A surviving monomial $a^{2i}b^{2j}$ becomes $y^i(1-y)^j$ with $i + j \leq Q/2$. Thus p is a polynomial of the asserted degree. For every $y \in [0,1]$, all four circuits are unitary, so $|p(y)| \leq 1$. When $y \leq s/T$, the entire one-dimensional system is promised and its target evolution is the phase $\mathrm{e}^{-\mathrm{i}Ty}$. The same query circuit must approximate this phase for all four encodings. Taking the return amplitude in Eq. (1.4) and averaging proves the second bound in Eq. (5.1). $\square$

5.2 Derivative and finite-difference estimates

Lemma 5.2 (Derivative estimate)**.** *If a complex polynomial p has degree $n \geq 1$ and satisfies $|p(x)| \leq 1$ on $[0,1]$, then, for $1 \leq k \leq n$,*

$$\sup_{x \in [0,1]} |p^{(k)}(x)| \leq \left(\frac{\mathrm{e}^2 n^2}{k}\right)^k. \tag{5.4}$$

Proof. Put $q(w) = p((w+1)/2)$. The expression

$$G(\zeta) = \zeta^n q\left(\frac{\zeta + \zeta^{-1}}{2}\right)$$

is a polynomial, including at $\zeta = 0$, and has modulus at most one on the unit circle. The maximum-modulus principle, applied at $\zeta = 1/v$, therefore gives

$$\left| q\left(\frac{v+v^{-1}}{2}\right) \right| \le |v|^n \qquad (|v| \ge 1). \tag{5.5}$$

Fix $x \in [0,1]$ and $|z-x| \le \rho$. Choose $|v| \ge 1$ with $2z-1 = (v+v^{-1})/2$, and write $\eta = \log|v|$. The focal identity for this parametrization is

$$\cosh\eta = |z| + |z-1| \le 1 + 2\rho.$$

Since $\cosh\eta \ge 1 + \eta^2/2$, we have $\eta \le 2\sqrt{\rho}$. Equation (5.5) gives $|p(z)| \le \mathrm{e}^{2n\sqrt{\rho}}$. Cauchy's estimate with $\rho = (k/n)^2$, followed by $k! \le k^k$, now yields

$$|p^{(k)}(x)| \le \frac{k!\mathrm{e}^{2n\sqrt{\rho}}}{\rho^k} \le \left(\frac{\mathrm{e}^2 n^2}{k}\right)^k.$$

□

Proposition 5.3 (Finite-parameter lower bound)**.** *Let p satisfy Eq. (5.1), and let $k \ge 1$ be an integer. Set $\nu = \min(s/k, 1)$. If*

$$\epsilon \le (\nu/8)^k, \tag{5.6}$$

then every associated simulator has

$$Q \ge \frac{\sqrt{Tk}}{\mathrm{e}}. \tag{5.7}$$

Proof. Let $h = \nu/T$. The points $0, h, \dots, kh$ lie in $[0, s/T]$. For the forward difference $\Delta_h^k f(0) = \sum_{j=0}^k (-1)^{k-j}\binom{k}{j} f(jh)$,

$$\left| \Delta_h^k \mathrm{e}^{-\mathrm{i}Ty}\big|_{y=0} \right| = |\mathrm{e}^{-\mathrm{i}\nu} - 1|^k = (2\sin(\nu/2))^k \ge (\nu/2)^k.$$

The difference between this quantity and $\Delta_h^k p(0)$ has modulus at most $2^k\epsilon \le (\nu/4)^k$. It follows that

$$|\Delta_h^k p(0)| \ge (\nu/4)^k. \tag{5.8}$$

Repeated use of the fundamental theorem of calculus gives

$$\Delta_h^k p(0) = \int_{[0,h]^k} p^{(k)}(u_1 + \cdots + u_k)\, du_1 \cdots du_k.$$

In particular, $\deg p = n \ge k$ and $\sup_{[0,1]} |p^{(k)}| \ge (T/4)^k$. Lemma 5.2 gives $n \ge \sqrt{Tk}/(2\mathrm{e})$. Since $n \le Q/2$, Eq. (5.7) follows. □

Proof of the lower bound in Theorem 1.1. Take

$$k = \max\left\{1, \left\lfloor \frac{L}{4D} \right\rfloor\right\}. \tag{5.9}$$

For every $L/D > 0$, this choice satisfies $k \ge L/(8D)$. If $k = 1$, Eq. (5.6) holds eventually because $\epsilon = o(s)$ and $\epsilon \to 0$ imply $\epsilon \le \min(s,1)/8$.

If $k \ge 2$, then eventually $k \le L$ and $\max(k/s, 1) \le 2 + L/s$. Since $D \to \infty$,

$$\begin{aligned} k\log(8\max(k/s,1)) &\le \frac{L}{4D}(\log 8 + D) \\ &\le L \end{aligned}$$

eventually. Exponentiating proves Eq. (5.6) in this case as well. Lemma 5.1 and Proposition 5.3 now imply

$$Q \geq \frac{1}{\mathrm{e}}\sqrt{\frac{TL}{8D}},$$

which matches the upper bound. The argument allowed arbitrary oracle-independent gates, workspace, and controlled or inverse queries. □

6 Scope and interpretation

The theorem answers the precision question for the stated factor-oracle model throughout Eq. (1.6). The construction works for an arbitrary exact encoding of an arbitrary square contraction; it does not assume that A is Hermitian or that its encoding has a preferred dilation. The only use of the low-energy promise is in evaluating the approximation error. Global contractivity makes the implementing circuit unitary on the full Hilbert space.

Three restrictions determine the meaning of the result. First, factor access is part of the input model: access to a block encoding of H alone is a different resource. Second, Eq. (1.4) fixes the output phase. The scalar lower-bound argument uses this convention and does not by itself establish an optimality statement for errors defined only on density operators. Third, query complexity leaves both classical preprocessing and oracle-independent gates uncharged. In particular, the exact factorization in Lemma 3.1 is not a bound on finite-precision compilation cost. Lower bounds for access to individual Hamiltonian terms, such as those of Zlokapa, Allen, and Harrow [4], address a different cost model and error convention.

The constants in the constructive bound were chosen to make the proof uniform, including families with bounded L/D. They are not intended as practical resource estimates. The theorem also assumes an exact factor oracle; its statement contains no robustness guarantee for an imperfect block encoding. Determining gate costs, handling oracle error, and comparing phase-insensitive error measures require further analysis beyond the result proved here.

A Verification record and manuscript provenance

The source package accompanying this draft contains the polynomial construction, deterministic diagnostic code, and a claim-to-evidence registry linked to the archived research record from round 23 of the QIQC project. The archived polynomial diagnostic evaluated 18 parameter choices on 103 points each at 160 decimal digits, for 1,854 evaluations. Those finite checks corroborate the formulas but do not prove global contractivity or asymptotic optimality. The analytical arguments in the main text supply those claims. The manuscript preparation reruns the polynomial diagnostic in a separate output directory and checks the walk identities on general complex contractions, unitary degree stripping, and four-sign polynomialization on finite matrix examples.

AI systems contributed to the research arguments, numerical diagnostics, internal reviews, source checking, and preparation of this manuscript. The coordinating assistant and the separate model review contexts were all AI components; their work is not reported as human mathematical verification. The author directed the research workflow and requested this manuscript. The K-Dense scientific-writing skill, from *Scientific Agent Skills* [5], guided evidence tracking and revision. Its use does not certify the proof.

This version remains a research draft awaiting final author review. The internal checks do not constitute independent expert confirmation, and the source comparison does not establish exhaustive publication priority.

References

[1] Alexander Zlokapa and Rolando D. Somma, “Hamiltonian simulation for low-energy states with optimal time dependence,” *Quantum* **8**, 1449 (2024). doi:10.22331/q-2024-08-27-1449. arXiv:2404.03644.

[2] Jeongwan Haah, “Product Decomposition of Periodic Functions in Quantum Signal Processing,” *Quantum* **3**, 190 (2019). doi:10.22331/q-2019-10-07-190. arXiv:1806.10236.

[3] Quantum Information and Quantum Computation Open Problem Zoo, “Optimal precision dependence of low-energy Hamiltonian simulation,” problem `op_25d9dee5435ea835`, accessed 21 September 2026. https://qiqc-op.com/problem/op_25d9dee5435ea835/.

[4] Alexander Zlokapa, Richard R. Allen, and Aram W. Harrow, “Optimal Lower Bounds for Hamiltonian Simulation,” arXiv:2607.19852 (2026).

[5] Timothy Kassis, Vinayak Agarwal, Yuhuan He, Darshil Patel, and Aubrey M. Brueckner, “Scientific Agent Skills: A Library of Procedural Knowledge for Research Agents,” arXiv:2609.00065 (2026). doi:10.48550/arXiv.2609.00065.