Convergence of the Uniformly Initialized Ježek–Řeháček–Fiurášek Iteration

QIQC research draft

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Working draft with internal computational and model review; external expert validation and novelty remain unverified.

Abstract

We prove convergence of the uniformly initialized JRF iteration for every finite quantum-state ensemble, allowing rank deficiency and overlapping supports. Square support compressions give a strongly convex objective and exact linear maximization over a fixed compact convex semialgebraic set. A Kurdyka–Łojasiewicz theorem yields compressed convergence without a prior lower bound on the normalizers. Determinant growth excludes a singular limiting normalizer, and noncommuting product expansion gives dual feasibility. Complementarity proves global optimality: the last POVM converges in the sum of Hilbert–Schmidt norms and its success probability converges to the optimum.

1 Statement and proof strategy

The JRF iteration is a nonlinear method for minimum-error quantum-state discrimination [1]. Monotonicity and directional or polar formulations are established ingredients [2, 3, 4]. Convergence for pure-state ensembles is treated by Lü and Dong [5]. The question considered here is convergence of the *specified last iterates* for arbitrary mixed-state ensembles, as posed in the QIQC problem collection [7].

All spaces below are finite dimensional. An asterisk denotes the adjoint. We use the Frobenius (Hilbert–Schmidt) norm $\|\cdot\|_{\mathrm{F}}$ and operator norm $\|\cdot\|_{\mathrm{op}}$. Complex matrix spaces are viewed as real Euclidean spaces with inner product $\langle U, V\rangle = \operatorname{Re}\operatorname{Tr}(U^*V)$. For tuples of matrices, the inner products and squared Frobenius norms are summed over components. A POVM is a tuple of positive semidefinite operators whose sum is the identity.

Theorem 1. *Let $m \geq 2$, and let $A_1, \dots, A_m$ be nonzero positive semidefinite operators on a finite-dimensional complex Hilbert space $\mathcal{H}$, and assume $\sum_i A_i \succ 0$. Initialize $\Pi_{i,0} = I/m$ and define*

$$D_k = \left(\sum_{i=1}^{m} A_i \Pi_{i,k} A_i\right)^{1/2}, \tag{1.1}$$

$$\Pi_{i,k+1} = D_k^{-1} A_i \Pi_{i,k} A_i D_k^{-1}. \tag{1.2}$$

Every finite D_k is invertible. There exists an optimal POVM $(\Pi_{i,\infty})_{i=1}^m$ such that

$$\sum_{i=1}^{m} \|\Pi_{i,k} - \Pi_{i,\infty}\|_{\mathrm{F}} \longrightarrow 0, \qquad \sum_{i=1}^{m} \operatorname{Tr}(A_i \Pi_{i,k}) \longrightarrow P_{\mathrm{opt}}, \tag{1.3}$$

where

$$P_{\mathrm{opt}} = \max_{\substack{Q_i \succeq 0 \\ \sum_i Q_i = I}} \sum_i \operatorname{Tr}(A_i Q_i).$$

For an ensemble $A_i = p_i\rho_i$, with $p_i > 0$ and $\sum_i p_i = 1$, take $\mathcal{H} = \operatorname{supp}(\sum_i p_i\rho_i)$. The hypothesis on the sum is then automatic. The theorem imposes no individual full-rank condition, support independence, simple-spectrum condition, or alteration of the iteration. Trace normalization is not needed in the proof. When $m = 1$, the unique POVM is I and the iteration fixes it, so the conclusion also covers single-state ensembles.

The proof has three separate tasks. First, a compressed sequence converges by a nonsmooth analytic convergence theorem. Second, a determinant recurrence makes its limiting normalizer positive definite and recovers convergence of the actual POVM. Third, bounded products establish dual optimality. In particular, convergence to a critical point will not be used as a substitute for global optimality.

2 Coherent factors and support compression

Choose support decompositions

$$A_i = V_i L_i V_i^*, \qquad V_i^* V_i = I_{r_i}, \qquad L_i \succ 0, \tag{2.1}$$

where $V_i : \mathbb{C}^{r_i} \to \mathcal{H}$ is an isometry onto $\operatorname{supp} A_i$. Starting with $X_{i,0} = I/\sqrt{m}$, use the *coherent* recurrence

$$X_{i,k+1} = X_{i,k} A_i D_k^{-1}. \tag{2.2}$$

The factors are not reset to principal square roots at later steps. Whenever the inverses exist, induction gives

$$X_{i,k}^* X_{i,k} = \Pi_{i,k}, \qquad X_k^* X_k = I, \quad X_k = (X_{1,k}; \dots ; X_{m,k}). \tag{2.3}$$

Define square matrices on the individual supports by

$$S_{i,k} = V_i^* X_{i,k} V_i, \qquad S_{i,0} = I_{r_i}/\sqrt{m}. \tag{2.4}$$

Lemma 2. *Every finite update is well defined and is a POVM. Every $S_{i,k}$ is invertible and has operator norm at most one. For all $k \geq 0$,*

$$Y_{i,k} := X_{i,k} A_i = V_i S_{i,k} L_i V_i^*, \tag{2.5}$$

$$S_{i,k+1} = S_{i,k} T_{i,k}, \qquad T_{i,k} = L_i V_i^* D_k^{-1} V_i. \tag{2.6}$$

Proof. The matrix $D_0^2 = m^{-1} \sum_i A_i^2$ is positive definite: a vector in its kernel is killed by every A_i and hence by their positive-definite sum. Equation (2.5) is direct at $k = 0$. For $k \geq 1$, the column range of $X_{i,k}$ lies in $\operatorname{ran} V_i$. Indeed this holds for $X_{i,1} = A_i D_0^{-1}/\sqrt{m}$ and is preserved by right multiplication. It implies $X_{i,k} V_i = V_i S_{i,k}$ and thus (2.5). Compressing (2.2) gives (2.6).

To close the induction, suppose $D_0, \dots, D_{k-1}$ are positive definite. Each compressed inverse $V_i^* D_j^{-1} V_i$ is positive definite, so (2.6) makes $S_{i,k}$ invertible. Equation (2.5) then gives

$$\ker Y_{i,k} = \ker V_i^* = \ker A_i.$$

The common kernel is zero, and hence $D_k^2 = \sum_i Y_{i,k}^* Y_{i,k} \succ 0$. Positivity is preserved by the literal update, and summing (1.2) gives the identity. This proves (2.3) and the POVM assertion at every finite step. Finally $\|X_{i,k}\|_{\mathrm{op}} \leq 1$ by (2.3), and compression by isometries gives $\|S_{i,k}\|_{\mathrm{op}} \leq 1$. □

The argument is only a finite-step induction. It does not assume a uniform bound on D_k^{-1}. The initial factor $X_{i,0}$ need not have column range in $\operatorname{ran} V_i$; the required initial identity was checked directly.

3 An exact iteration on a fixed convex set

Let $d = \dim \mathcal{H}$ and define the rectangular contraction ball and its linear image

$$\mathcal{B} = \{Z = (Z_1; \ldots; Z_m) : Z_i \in \mathbb{C}^{d\times d},\ Z^*Z \preceq I\}, \tag{3.1}$$

$$\mathcal{P}(Z) = (V_i^* Z_i V_i)_i, \qquad K = \mathcal{P}(\mathcal{B}). \tag{3.2}$$

The ball $\mathcal{B}$ is compact, convex and semialgebraic in real and imaginary coordinates. For example, positivity of $I - Z^*Z$ can be expressed by its principal minors. Its linear image K is compact and convex, and the projection theorem for semialgebraic sets gives semialgebraicity. Fixed real coefficients in these descriptions are allowed; the entries of V_i need not be algebraic numbers. By (2.3), $S_k := (S_{i,k})_i = \mathcal{P}(X_k)$ belongs to K.

On the full product of the support matrix spaces, put

$$f(S) = \sum_i \operatorname{Tr}(L_i S_i^* S_i), \qquad \mu = \min_i \lambda_{\min}(L_i) > 0, \qquad L_{\max} = \max_i \|L_i\|_{\mathrm{op}}. \tag{3.3}$$

This real quadratic polynomial has gradient $(2S_iL_i)_i$. Expansion gives

$$f(T) \geq f(S) + \langle \nabla f(S), T - S\rangle + \mu\|T - S\|_{\mathrm{F}}^2, \tag{3.4}$$

$$\|\nabla f(T) - \nabla f(S)\|_{\mathrm{F}} \leq 2L_{\max}\|T - S\|_{\mathrm{F}}. \tag{3.5}$$

Thus individual rank deficiency has disappeared from the strong-convexity constant: only positive support eigenvalues enter (3.3). For the actual iterates,

$$f(S_k) = \sum_i \operatorname{Tr}(A_i \Pi_{i,k}). \tag{3.6}$$

For $k \geq 1$ this follows from $X_{i,k}V_i = V_iS_{i,k}$; for $k = 0$ both sides equal $m^{-1}\sum_i \operatorname{Tr} A_i$.

Lemma 3. *The compressed literal update satisfies*

$$S_{k+1} \in \arg\max_{S\in K} \langle \nabla f(S_k), S\rangle. \tag{3.7}$$

Proof. For every $Z \in \mathcal{B}$, equation (2.5) gives the exact pullback identity

$$\langle \nabla f(S_k), \mathcal{P}(Z)\rangle = 2\operatorname{Re}\operatorname{Tr}(Y_k^* Z). \tag{3.8}$$

The stacked literal update is $X_{k+1} = Y_kD_k^{-1}$, the polar isometry of the full-column-rank matrix Y_k. Since $Y_k = X_{k+1}D_k$, any contraction Z satisfies

$$\operatorname{Re}\operatorname{Tr}(Y_k^* Z) = \operatorname{Re}\operatorname{Tr}(D_kX_{k+1}^* Z) \leq \operatorname{Tr} D_k.$$

Indeed X_{k+1}^*Z is a contraction, so in an eigenbasis of D_k each of its diagonal real parts is at most one. Equality holds at $Z = X_{k+1}$. Equation (3.8) and $K = \mathcal{P}(\mathcal{B})$ prove the claim. □

The set K is fixed. Overlapping supports remain coupled through the one stacked contraction constraint in (3.1); no product of independent support balls has replaced it. Equation (3.7) maximizes a linear functional, not the nonlinear objective f.

4 Convergence of the compressed sequence

We state the convergence input in the form needed here. The subdifferential ∂h denotes the limiting subdifferential. A proper lower semicontinuous semialgebraic function has the Kurdyka–Łojasiewicz (KL) property: near a point $\bar{z}$ in its subdifferential domain, a continuous concave function φ, with $\varphi(0) = 0$ and $\varphi' > 0$ on a small positive interval, satisfies

$$\varphi'(h(z) - h(\bar{z}))\ \operatorname{dist}(0, \partial h(z)) \geq 1 \quad \text{when } h(\bar{z}) < h(z) < h(\bar{z}) + \eta.$$

The following is the abstract descent theorem of Attouch–Bolte–Svaiter [6, Theorem 2.9].

Theorem 4 (KL descent criterion)**.** *Let h be proper and lower semicontinuous on a finite-dimensional real Euclidean space. Suppose a sequence z_k has constants $a, b > 0$ such that*

$$\begin{aligned}
&\text{(H1)} \quad h(z_{k+1}) + a\|z_{k+1} - z_k\|^2 \le h(z_k),\\
&\text{(H2)} \quad \textit{some } w_{k+1} \in \partial h(z_{k+1}) \textit{ satisfies } \|w_{k+1}\| \le b\|z_{k+1} - z_k\|,\\
&\text{(H3)} \quad z_{k_j} \to \bar{z} \quad \textit{and} \quad h(z_{k_j}) \to h(\bar{z}) \textit{ for a subsequence.}
\end{aligned}$$

If h has the KL property at $\bar{z}$, the whole sequence converges to $\bar{z}$, this point is critical, and $\sum_k \|z_{k+1} - z_k\| < \infty$.

Apply the theorem to

$$g = -f + \iota_K, \qquad \iota_K(S) = \begin{cases} 0 & S \in K, \\ +\infty & S \notin K. \end{cases} \tag{4.1}$$

This function is proper, lower semicontinuous and semialgebraic; it is bounded below because K is compact. For a closed convex set, the normal cone is

$$N_K(S) = \{W : \langle W, T - S \rangle \le 0 \text{ for every } T \in K\}.$$

Its convex and limiting versions coincide. Smooth perturbation of the indicator gives $\partial g(S) = -\nabla f(S) + N_K(S)$ on K.

First, equations (3.4) and (3.7) imply

$$g(S_{k+1}) + \mu\|S_{k+1} - S_k\|_{\mathrm{F}}^2 \le g(S_k). \tag{4.2}$$

Second, the exact maximization implies $\nabla f(S_k) \in N_K(S_{k+1})$. Therefore

$$w_{k+1} := \nabla f(S_k) - \nabla f(S_{k+1}) \in \partial g(S_{k+1}), \tag{4.3}$$

$$\|w_{k+1}\|_{\mathrm{F}} \le 2L_{\max}\|S_{k+1} - S_k\|_{\mathrm{F}}. \tag{4.4}$$

The subgradient in (4.3) is at the *next* iterate, as required by (H2). Third, compactness supplies a subsequence $S_{k_j} \to \bar{S} \in K$. Polynomial continuity gives $g(S_{k_j}) = -f(S_{k_j}) \to -f(\bar{S}) = g(\bar{S})$, verifying (H3). The KL property applies on K; the subdifferential is nonempty there because $0 \in N_K(S)$.

All hypotheses now hold, for instance for the sequence starting at $k = 1$. Consequently

$$S_k \longrightarrow S_\infty \in K, \qquad \sum_{k=1}^{\infty} \|S_{k+1} - S_k\|_{\mathrm{F}} < \infty. \tag{4.5}$$

No lower bound on D_k was used. Theorem 4 supplies convergence to a critical point only; the optimality proof remains to be given.

5 Excluding a singular limiting normalizer

The compressed identity (2.5) implies

$$D_k^2 = \sum_i V_i L_i S_{i,k}^* S_{i,k} L_i V_i^*. \tag{5.1}$$

Equation (4.5) and continuity of the positive square root therefore give $D_k \to D_\infty \succeq 0$. Also $0 \preceq \Pi_{i,k} \preceq I$ implies $D_k^2 \preceq \sum_i A_i^2$, so fix $C > 0$ with $D_k \preceq CI$ for all k.

Lemma 5. *The limit D_∞ is positive definite.*

Proof. Suppose u is a unit vector in $\ker D_\infty$. Because the supports span $\mathcal{H}$, there is a fixed index i with $z = V_i^* u \neq 0$. Put $x = V_i z$; then $x^* u = \|z\|^2$. Weighted Cauchy–Schwarz gives

$$x^* D_k^{-1} x \geq \frac{|x^* u|^2}{u^* D_k u} = \frac{\|z\|^4}{u^* D_k u}. \tag{5.2}$$

The denominator is positive at each finite step and tends to zero. For $C_{i,k} = V_i^* D_k^{-1} V_i$, the Rayleigh quotient at z thus gives

$$\lambda_{\max}(C_{i,k}) \geq \frac{\|z\|^2}{u^* D_k u} \longrightarrow \infty.$$

Every eigenvalue of $C_{i,k}$ is at least $1/C$. Its fixed dimension r_i therefore yields

$$\det C_{i,k} \geq C^{-(r_i-1)} \lambda_{\max}(C_{i,k}) \longrightarrow \infty, \qquad \det T_{i,k} = \det L_i \det C_{i,k} \longrightarrow \infty. \tag{5.3}$$

This includes $r_i = 1$. Taking determinant moduli in (2.6) gives

$$|\det S_{i,k+1}| = |\det S_{i,k}| \det T_{i,k}. \tag{5.4}$$

All finite determinants on the left are nonzero. The multiplier is eventually at least two, so (5.4) forces their moduli to grow without bound. This contradicts $\|S_{i,k}\|_{\mathrm{op}} \leq 1$. □

We can now recover the literal factors without any continuity issue for inversion. Equations (2.5), (4.5) and Lemma 5 show that

$$X_{i,k+1} = V_i S_{i,k} L_i V_i^* D_k^{-1} \longrightarrow X_{i,\infty}.$$

Thus the whole factor sequence converges, as do the effects $\Pi_{i,k} = X_{i,k}^* X_{i,k}$ to $\Pi_{i,\infty} = X_{i,\infty}^* X_{i,\infty}$. Their sum is the identity, and the first limit in (1.3) follows. The order of the argument matters: convergence of D_k was proved from compressed convergence before its possible singularity was excluded.

6 Noncommuting products and dual feasibility

The remaining task is to prove $D_\infty \succeq A_i$ for every outcome. The following product fact supplies the required obstruction to a nonoptimal limit.

Lemma 6 (Expanding products)**.** *Suppose invertible square matrices T_k converge to a matrix T similar to a positive-definite Hermitian matrix. Suppose also that invertible square matrices S_k are uniformly bounded and obey $S_{k+1} = S_k T_k$. Then every eigenvalue of T is at most one.*

Proof. If the largest eigenvalue is $\lambda > 1$, choose one fixed similarity diagonalizing T^*. Separate its entire λ-eigenspace E from the remaining subspace F. In these coordinates the limit is $\mathrm{diag}(\lambda I_E, H_F)$. If $F \neq \{0\}$, write $\nu = \|H_F\|_{\mathrm{op}} < \lambda$. For all sufficiently late k, the transformed T_k^* differs from this limit by norm at most ε, where

$$\lambda - 2\varepsilon > 1, \qquad \nu + 2\varepsilon < \lambda - 2\varepsilon.$$

If $\|v_F\| \leq \|v_E\|$, the next E component has norm at least $(\lambda - 2\varepsilon)\|v_E\|$, while the next F component has norm at most $(\nu + 2\varepsilon)\|v_E\|$. Hence this cone is invariant. A nonzero vector initially in E grows exponentially under the reverse products $T_{k-1}^* \cdots T_n^*$ for a sufficiently late, fixed n. If $F = \{0\}$, all sufficiently late transformed factors have minimum singular value greater than one, yielding the same conclusion.

Taking adjoints, the forward products $T_n \cdots T_{k-1}$ have unbounded norm. But the recurrence gives the exact order

$$T_n T_{n+1} \cdots T_{k-1} = S_n^{-1} S_k, \tag{6.1}$$

which is bounded for fixed n. A fixed invertible similarity changes norm bounds only by a constant. This contradiction proves the lemma. Using the entire top eigenspace covers eigenvalue multiplicity; no commuting-factor assumption has entered the proof. □

Apply Lemma 6 separately to every recurrence (2.6). The limits are

$$T_i = L_i V_i^* D_\infty^{-1} V_i, \tag{6.2}$$

each similar to the positive-definite Hermitian matrix

$$H_i = L_i^{1/2} V_i^* D_\infty^{-1} V_i L_i^{1/2}. \tag{6.3}$$

Boundedness and invertibility of $S_{i,k}$ were proved in Lemma 2. The product lemma implies $H_i \preceq I_{r_i}$. Writing $R_i = D_\infty^{-1/2} V_i L_i^{1/2}$, we obtain

$$\begin{aligned} H_i = R_i^* R_i \preceq I_{r_i} &\iff R_i R_i^* \preceq I_{\mathcal{H}} \\ &\iff A_i \preceq D_\infty. \end{aligned} \tag{6.4}$$

The first equivalence follows from the common nonzero singular spectrum, or from the operator norm of R_i; the second is congruence by $D_\infty^{1/2}$. Only the square support compression $S_{i,k}$ was inverted, not a rank-deficient factor on the full signal space.

7 Complementarity and global optimality

Taking limits in $X_{i,k+1} D_k = X_{i,k} A_i$ gives $X_{i,\infty} D_\infty = X_{i,\infty} A_i$. Taking adjoints and multiplying on the right by $X_{i,\infty}$ yields complementarity:

$$(D_\infty - A_i)\Pi_{i,\infty} = 0. \tag{7.1}$$

Summing and using $\sum_i \Pi_{i,\infty} = I$ gives

$$D_\infty = \sum_i A_i \Pi_{i,\infty}, \qquad \operatorname{Tr} D_\infty = \sum_i \operatorname{Tr}(A_i \Pi_{i,\infty}). \tag{7.2}$$

For any competing POVM (Q_i), dual feasibility (6.4) implies

$$\sum_i \operatorname{Tr}(A_i Q_i) \le \sum_i \operatorname{Tr}(D_\infty Q_i) = \operatorname{Tr} D_\infty. \tag{7.3}$$

The limiting POVM attains this upper bound by (7.2), so it is globally optimal. Continuity of the score gives the second limit in (1.3). This completes the proof of Theorem 1. □

A Scope and provenance

This exposition preserves the scope of the frozen round-10 general JRF claim. It concerns finite ensembles on their joint signal support and the uniform initial POVM. It makes no assertion about arbitrary nonuniform initialization, an ensemble-independent convergence rate, or infinite-dimensional systems. The external convergence theorem is stated with its hypotheses; the determinant and product arguments separately establish the conclusions it does not supply.

The project record contains an independent argument audit and a separate model review of the frozen complete proof, recorded as `PASS / exact / full_resolution`. Numerical diagnostics checked finite matrix identities; none is a premise of the universal proof. The canonical frozen claim has SHA-256 fingerprint

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a64eba4ab6ddc6f8d52b98427ee7f67d

The discussion of the prior Stiefel formulation in [4] refers to the accepted manuscript inspected in the project. Attribution here does not constitute an exhaustive novelty search.

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[7] QIQC Open Problems, "Convergence of the JRF iteration for mixed-state discrimination," problem `op_76e284219621a785`. https://qiqc-op.com/problem/op_76e284219621a785/. Problem statement frozen for the project on 15 September 2026.