One-shot coherence and deterministic work costs for Gibbs-preserving channels

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Abstract

Implementing a channel that preserves a Gibbs state can require a clock, a battery, or both. For the two separate resource tasks posed by Tajima and Takagi and formalized in the QIQCOP Zoo, we give channel-by-channel variational characterizations. At positive channel error, finite semidefinite programs bound the coherence cost from both sides. At zero error, matrix positivity on the energy lattice characterizes finite-clock feasibility. The first inconsistent joint moment equation in a minimal Stinespring representation fixes the power of the divergence. For deterministic work, a sharp battery imposes a covariance threshold. Above or at that threshold the answer depends on which thermal processes are admitted. For the same three-level channel at inverse temperature one, finite baths give infinite cost, a normal Gibbs bath gives $\log 16$, and closing the joint operations before imposing the battery endpoint gives $\log(529/36)$. The detailed proofs and computational controls accompany this internally reviewed proof candidate.

1 Two resource tasks

Let A, B be finite systems with Hamiltonians H_A, H_B. At inverse temperature β write $\gamma_X = e^{-\beta H_X}/Z_X(\beta)$, where $Z_X(u) = \operatorname{Tr} e^{-uH_X}$. A channel $\Phi : A \to B$ preserves the Gibbs state if $\Phi(\gamma_A) = \gamma_B$. We measure implementation error by the supremum of purified distance over inputs on the system and a reference. This choice tests the survival of correlations with another system. A thermal operation appends systems in their Gibbs states, applies a unitary conserving the total Hamiltonian and discards the designated output environment.

In the *coherence* task a finite resource with arbitrary state and Hamiltonian may be supplied. Its charge is its symmetric logarithmic derivative quantum Fisher information (QFI), normalized to $4\operatorname{Var}(H)$ on pure states. A sharp-energy ancillary register may be included at zero coherence charge. Write $F_{\mathrm{c}}^{\varepsilon}(\Phi)$ for the infimum over implementations within error ε.

In the *work* task the battery supplies the only nonthermal input. It starts and ends in prescribed pure energy eigenstates for every system input. Its energy loss $W = w_i - w_f$ is the deterministic work. We compare finite Gibbs baths, countable baths with a trace-class Gibbs state, and limits of finite thermal operations on the entire system and battery at a fixed gap. System approximation and sharpness of the battery endpoint are separate requirements. The work formulas in the main text use $0 < \beta < \infty$. The supplement treats zero and negative inverse temperatures and specifies the lower-bound convention for bath Hamiltonians.

Tajima and Takagi established infinite-coherence examples and a quantitative obstruction for pairwise reversible channels [2]. Faist, Berta and Brandão derived a universal many-copy work rate and one-shot implementations for important time-covariant cases [4]. The question here asks for each individual channel at each error, including exact implementation [1].

2 Coherence from energy-sector isometries

Fix input and output energy eigenbases with energies e_i, f_j, retaining all degeneracies. Use the unnormalized input-first Choi matrix J_Φ. The variables are a finite probability distribution p on clock energies and finitely supported positive Choi matrices Y_z indexed by environment energy. The full conservation and output conditions are

$$\sum_j (Y_{g-f_j})_{ij,i'j} = \delta_{ii'} p_{g-e_i} \quad (g \in \mathbb{R}), \qquad J = \sum_z Y_z. \tag{1}$$

Unoccupied energies carry zero weight. A finite pure clock realizing J exists with energy law p exactly when (1) holds.

To see this, take a covariant Stinespring dilation and group its Kraus vectors by final environment energy. In each total-energy sector the column Gram matrix of the weighted isometry is the diagonal matrix on the right of (1). Conversely, factor each Y_z and assemble the corresponding row vectors. Divide each occupied column by the square root of its input mass. The resulting columns are orthonormal, so finite sharp blank registers permit completion to an energy-conserving unitary. Inserting the clock with amplitudes $\sqrt{p_x}$ recovers the required Kraus map. Marvian's optimized-purification identity [3, Theorem 3] gives the same optimum over mixed resources. Translating the clock spectrum centers p without changing its variance.

For a finite grid, constrain the approximation by an auxiliary matrix X:

$$\begin{pmatrix} J_\Phi & X \\ X^* & J \end{pmatrix} \succeq 0, \qquad \operatorname{Re} \operatorname{Tr}_B X \succeq \sqrt{1-\eta^2}\, \mathbf{1}_A. \tag{2}$$

Here $\operatorname{Re} M = (M + M^*)/2$. The state fidelity block program followed by minimization over input marginals gives exactly the reference-assisted channel constraint. Choose rationally independent generators ω for the finite energy lattice after shifting energy origins. Write $e_i = \omega \cdot a_i,\ f_j = \omega \cdot b_j$ with integral labels. If all system energies are degenerate the coherence cost is zero. Otherwise let q be the lattice rank, $d = \dim A$, $Q = \max_{i,j} \|b_j - a_i\|_1$, $B_* = \max_{i,j} \|b_j - a_i\|_\infty$, and $\Omega = \sum_\ell |\omega_\ell|$. Put

$$K_p = \lceil NL(1 + \Omega/|\omega_1|) \rceil + 1, \qquad K = K_p + NB_*, \qquad \delta_L = \min\{1, \sqrt{2dQ/L}\}.$$

Use clock and environment indices in $[-K, K]^q$ and grid energies $\omega \cdot z/N$. Let $C_{L,N}(\eta)$ minimize $4 \sum_x x^2 p_x$ subject to the sector and channel constraints on the grid. At $\eta \geq 1$ drop the accuracy constraint. Then

$$\max\{0, C_{L,N}(\varepsilon + \delta_L) - \omega_1^2/N^2\} \ \leq \ F_{\mathrm{c}}^{\varepsilon}(\Phi) \ \leq \ C_{L,N}(\varepsilon). \tag{3}$$

The bounds converge together for $0 < \varepsilon < 1$ along $L = h,\ N = h!$. Localizing a clock in translated lattice boxes dephases it by at most δ_L in channel distance. Recentering the boxes on the grid costs at most ω_1^2/N^2. Convexity in required root fidelity supplies continuity at interior errors. A free covariant implementation suffices whenever the error ball meets the compact set of covariant channels. The grid and convergence proofs are in the supplement under *Finite semidefinite brackets*.

3 The endpoint and the small-error law

3.1 A finite exact clock

In a minimal Kraus basis let $T \succ 0$ be the normalized target Gram, equal to the identity when the Kraus vectors are used directly as columns. Since $0 \preceq Y_z \preceq J_\Phi$ at zero error, the variables are supported on this Kraus space. Write their relative Grams C_z in the conjugate convention $Y_z = BC_z^{\mathsf{T}}B^*$, where B collects the Kraus vectors. Encode the sector equations for p, C as linear Laurent polynomial identities on $\mathbb{Z}^q$, and impose $C(1) = p(1)T$. A finite exact implementation exists precisely when this module contains a block diagonal element $\mathrm{diag}(p, C)$ with positive-semidefinite coefficients and positive-definite sum. Normalize by its total scalar mass. A physical clock can occupy several translates of the lattice. Translate and add all its positive tuples before imposing the terminal Gram constraint.

The finite criterion uses positive torus evaluations and boundary directions. At each $x \in (0,\infty)^q$ the evaluated module must contain a positive-definite matrix. For each exterior normal there must be a local member with a descending flag of surviving column spaces. Its leading diagonal coefficient blocks are positive semidefinite, their sum on each successive quotient is positive definite, and their kernels annihilate the corresponding cross columns. Initial-module generators reduce each boundary test to positive feasibility on a lattice of strictly smaller rank. A finite sphere cover and a positive scalar multiplier combine the local witnesses. The base case is ordinary matrix positivity. This gives a terminating algebraic decision for algebraic input entries with exact energy-lattice labels supplied. The scalar initial-module input is due to Einsiedler, Mouat and Tuncel [5]; the matrix kernel condition and the tangential recursion are proved in the supplement.

When a finite exact seed exists, the infimum of $4\,\mathrm{Var}(p)$ in (1) has a matching lower dual. Let $G_j = \sum_a g_{ja} X^a$ be generators of the sector module before the terminal constraint, $s_a = \omega \cdot a$, $p(g)$ its scalar coordinate and $D(G_j) = \sum_a (C(g_{ja}) - Tp(g_{ja}))$. Maximize c over a Hermitian linear functional λ and positive scalar-plus-matrix fields

$$Y(x) = \sum_{t \in I} e^{itx}(A_t + xB_t + x^2 D_t) \succeq 0 \qquad (x \in \mathbb{R})$$

with finitely many real frequencies and conjugate pairing, subject to

$$\sum_a \langle g_{ja}, Y(x + s_a)\rangle = \sum_a ((x + s_a)^2 - c)p(g_{ja}) + \lambda(D(G_j)). \tag{4}$$

The supremum multiplied by four is $F_{\mathrm{c}}^0(\Phi)$. Pairing (4) with each translated physical tuple gives the lower bound. For the converse a finite positive seed majorizes the ambient ordered space. Positive functional extension, convolution, projection through the singular frequencies of the symbol, and positive trigonometric averaging give lower witnesses arbitrarily near the infimum. Positivity is required on the entire real axis. The two proofs *A sharp lower-bound dual* and *Finite lower witnesses* in the supplement treat this step, including physical resonances.

3.2 Which power of the error occurs?

Take a minimal Stinespring isometry $V : A \to B \otimes E$ and define $\Omega_j(X) = V^*(H_B^j \otimes X)V$. For each order m, solve *jointly* for Hermitian $C_1, \ldots, C_m$, with $C_0 = \mathbf{1}_E$:

$$\sum_{j=0}^{\ell} \binom{\ell}{j} \Omega_j(C_{\ell-j}) = H_A^\ell \quad (0 \le \ell \le m). \tag{5}$$

If k is the first failed order, then $F_{\mathrm{c}}^{\varepsilon}(\Phi) = \Theta_{\Phi}(\varepsilon^{-2/k})$ and the exact cost is infinite. If the equations remain consistent at every order, the small-positive-error cost stays bounded. In this branch the zero-error finite-clock criterion still has to be tested. With a, b distinct input and output energies and $r = \dim E$, it suffices to test through $a\binom{b+p-1}{p} - 1$, where $p = \min(r^2, d^2)$.

The proof takes the Taylor system of $V^*(e^{uH_B} \otimes C(u))V = e^{uH_A}$. Analytic row and column elimination identify the first obstruction and yield its variance lower bound. Translated finite clock distributions realizing the consistent moments yield the same upper exponent. In the bounded branch a finite-second-moment measure relaxation gives the limiting cost. The proofs of the lower bound, the finite construction, and the relaxation are separate in the supplement.

4 Sharp work

A stationary battery and a Gibbs bath induce a time-translation covariant channel. Let C be the covariant trace-preserving channels in the closed error ball about Φ. If C is empty the work is infinite. For normal Gibbs baths at positive temperature the converse holds, including when the ball first touches the covariant set.

Here is the variational value when $C \neq \varnothing$. Subtract the individual system ground energies, and let S be the span of the Kraus supports of C. Choose a basis K_a of S with frequencies ν_a, and use the Gram convention $\Psi_T(X) = \sum_{a,b} T_{ba} K_a X K_b^*$. Choose any finite nonzero list of lattice multiplicities m_n, put $\mathcal{E}_n = \mathbb{C}^{m_n}$, and at output grade k take a positive matrix G^k on $\bigoplus_a \mathcal{E}_{k+\nu_a}$. Its blocks obey the isometry conditions

$$\sum_{\substack{k,a,b \\ n=k+\nu_a,\ n'=k+\nu_b}} K_a^* K_b \otimes G_{ab}^k = \delta_{nn'} \mathbf{1}_A \otimes \mathbf{1}_{\mathcal{E}_n}. \tag{6}$$

For $u \geq \beta$, set $[L_u(G)]_{ab} = \sum_k e^{-u(k+\nu_a)} \operatorname{Tr} G_{ab}^k$ on equal frequencies and zero otherwise. Let $\mathcal{H}_u(S)$ contain those Hermitian functionals H for which every such finite list satisfies

$$\langle H, L_u(G) \rangle \ \leq \ \frac{Z_B(u)}{Z_A(u)} \sum_k e^{-uk} \operatorname{rank} G^k. \tag{7}$$

Write e for the functional equal to one on normalized trace-preserving Grams. Define $d_u(S) = \sup\{z : ze \in \mathcal{H}_u(S)\}$, and

$$b_C = \sup_{H \in \mathcal{H}_\beta(S)} \min_{T \in C} \langle H, T \rangle.$$

Let $g(S)$ be the smallest possible maximal Kraus frequency of a covariant trace-preserving channel supported in S. Testing successive frequency cutoffs computes $g(S)$. The work is

$$W_{\varepsilon}^{\mathrm{normal}}(\Phi) = \max\left\{ g(S),\ \sup_{u \geq \beta} \frac{\log d_u(S)}{u},\ \frac{\log b_C}{\beta} \right\}. \tag{8}$$

Factoring G^k gives an isometry with minimal output dimension $\operatorname{rank} G^k$ in that grade. Its Gibbs weights give precisely the trace and rank in (7). A finite average of elements of C has the full support S and still satisfies the error bound. Convex separation on this face gives b_C, including a ball tangent to the covariant set. The ground and all-temperature terms express the additional support constraints for an exactly sharp physical implementation. A normal Gibbs bath completes the finite logical isometries at every work value strictly above the infimum. The dimension quantifier

in (7) matters. The supplement gives an undecidability reduction for a universal positive-cost test on rational full-rank unital channels of varying dimension, using [7, 6].

For actual finite baths, take a full-rank input and the faithful bath Gibbs state. A sharp deterministic battery transition restricts the unitary to an isometry between the full finite nonbattery spaces. Their spectra must agree after translating by W, so their largest energies force $W = 0$. Thus the work is zero if a finite thermal channel meets the closed error ball and infinite otherwise. The finite multiplicity and unitary test is given in the supplement. If an exact-infeasible target belongs to the closure of the finite thermal channels, its work is zero at every positive error. Outside this closure the work is infinite throughout a neighborhood of zero.

One can instead take the norm closure of the *joint* finite thermal operations at a fixed battery gap, and impose a pure final battery only on the limit. This convention is discussed following Eq. (S.16) of [2]. With $C \neq \varnothing$ the work in this model is

$$W_\varepsilon^{\mathrm{closed}}(\Phi) = \frac{1}{\beta} \log \sup_{H \in \mathcal{H}_\beta(\mathcal{L}(A,B))} \min_{T \in C} \langle H, T \rangle. \tag{9}$$

For a finite approximant the battery success block can be a contraction. In each energy shell its trace defect bounds both the missing output dimension and the mean squared polar correction. Adding just these missing vectors gives a genuine logical isometry at a partition cost tending to the original gap. Conversely full-support approximations admit a common fixed gap above the right side of (9). Truncating their normal baths while retaining the physical spectral width yields convergent whole processes.

5 Two channels with explicit costs

A three-level channel. Set $\beta = 1$, $h = \log 4$, $H_A = h\,\mathrm{diag}(0, 1, 3)$, $H_B = h\,\mathrm{diag}(0, 2, 3)$. In matrix-unit notation $E_{ij} = |i\rangle\langle j|$, starting at zero, let

$$K_0 = \sqrt{a}E_{00} + \sqrt{c}E_{22}, \qquad K_+ = \sqrt{1-a}E_{10} + \sqrt{b}E_{21}, \qquad K_- = \sqrt{1-b}E_{01} + \sqrt{1-c}E_{12},$$

where $a = 2751/2944$, $b = 31/736$, $c = 1/2$. These Kraus operators preserve trace and send $\mathrm{diag}(64, 16, 1)/81$ to $\mathrm{diag}(64, 4, 1)/69$. Their frequencies are $0, 2h, -h$. The nine products $K_i^* K_j$ are independent, so the channel is extreme. Writing $y = e^{-uh}$, its partition ratio is

$$R(y) = \frac{(1 + y^2 + y^3)^2}{y^2(1 + y + y^3)}.$$

At $u = 1$, $R(1/4) = 529/36$. For all $0 < y \leq 1/4$, the difference $1 + y + y^3 - (1 + y^2 + y^3)^2$ equals $y(1 - 2y - y^2 - y^3 - 2y^4 - y^5) > 0$. Thus $\log R(e^{-uh})/u < 2h$, with limit $2h = \log 16$ at large u. The extreme-channel formula and the fixed-gap closure theorem give the following costs for this single target.

Thermal process and battery condition	W_0 at $\beta = 1$
Finite bath, exactly sharp endpoint	$+\infty$
Normal Gibbs bath, exactly sharp endpoint	$\log 16$
Closed joint process, sharp endpoint in the limit	$\log(529/36)$

A qubit measurement. Take $H_A = |1\rangle\langle 1|$. Measure in the X basis and record the result in a degenerate classical output bit. Every input Gibbs state gives unbiased outcomes. The first moment system fails because $V^*(I \otimes C_1)V$ lies in the span of I, X. Hence the small-error coherence cost grows as ε^{-2}. A covariant channel maps $|+\rangle$ and $|-\rangle$ to the same output; the two target records are orthogonal, so one has purified distance at least $1/\sqrt{2}$. The maximally mixed replacement D attains that bound even with a reference. Indeed $2J_D \geq J_\Phi$, so for every input its output dominates half of the target state and the root fidelity is at least $1/\sqrt{2}$. A sharp battery alone therefore gives infinite work below that error and zero at the threshold.

Proofs, checks and review

The proofs and reproducibility package contains the proof of each variational equality, the finite-clock decision and the moment construction. Its reading guide identifies the individual arguments and the endpoint conventions. The executable checks cover finite examples, exact algebra and deliberately incorrect variants, including a transposed Gram convention. The general claims rest on the analytic proofs. The derivation and proof review were performed sequentially by the same AI researcher. Independent specialist review and priority remain open.

References

[1] QIQCOP Zoo, Resources for implementing Gibbs-preserving channels, problem record.

[2] H. Tajima and R. Takagi, Gibbs-preserving operations requiring infinite amount of quantum coherence, Phys. Rev. Lett. **134**, 170201 (2025).

[3] I. Marvian, Operational interpretation of quantum Fisher information in quantum thermodynamics, Phys. Rev. Lett. **129**, 190502 (2022).

[4] P. Faist, M. Berta and F. G. S. L. Brandão, Thermodynamic implementations of quantum processes, Commun. Math. Phys. **384**, 1709–1750 (2021).

[5] M. Einsiedler, R. Mouat and S. Tuncel, When does a submodule of $(\mathbb{R}[x_1, \ldots, x_k])^n$ contain a positive element?

[6] M. Musat and M. Rørdam, Non-closure of quantum correlation matrices and factorizable channels that require infinite dimensional ancilla.

[7] I. Goldbring and B. Hart, The universal theory of the hyperfinite II_1 factor is not computable.