A dimension-free lower bound on γ_3 for NPT bound entanglement

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Abstract

Bharti, Gajjala and Haug resolve the two-copy distillability problem for Werner states and construct explicit constants γ_k such that $\alpha \geq -\gamma_k$ implies k-copy undistillability in every dimension. For $k = 3$ we report $\gamma_3 \geq (1 + 3^{1/3} - 3^{2/3})/2 = 0.181083\ldots$, about 8.65% above the published value 1/6. Separately, exact minimisation over rank-two witnesses returns zero at every local dimension from two to six, which points to the much stronger $\gamma_3 = 1/2$ — that is, the third copy does not help at the endpoint. *The bound has not been adversarially verified*; it should be read as a lead.

1. Background

A state with non-positive partial transpose may still be undistillable, a possibility raised independently by DiVincenzo, Shor, Smolin, Terhal and Thapliyal [3] and by Dür, Cirac, Lewenstein and Bruß [4], and unresolved for twenty-five years. It appears as Problem 5 in the list of Horodecki, Rudnicki and Życzkowski [2].

In July 2026 the two-copy case fell. Bharti, Gajjala and Haug [1] prove a sharp dimension-free partial-trace inequality and deduce that a Werner state ρ_α is two-copy distillable if and only if $\alpha < -1/2$, settling Problem 5 of [2] at two copies. They also construct explicit constants $\gamma_k > 0$ with $\alpha \geq -\gamma_k$ implying k-copy undistillability in every dimension, which moves the live frontier to three copies. (Their own abstract records that the initial proofs were machine-generated and then verified and rewritten by the authors — the same division of labour as this note.)

2. The problem

Three-copy endpoint. *Determine the largest γ_3 such that $\alpha \geq -\gamma_3$ implies three-copy undistillability of ρ_α in every local dimension. Equivalently, at the endpoint $\alpha = -1/2$, decide whether the endpoint partial-trace form q_3 is nonnegative on every operator of rank at most two.*

The published value is $\gamma_3 = 1/6$ [1]; the two-copy answer is 1/2.

3. The bound

The route is a lemma bounding the three-term sum,

$$A_1 + A_2 + A_3 \;\leq\; 4N + T,$$

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obtained from a balanced-frame estimate together with the swap majorization $F_1+F_2+F_3 \prec 2I + F_1F_2F_3$. This gives $q_3(-t) \geq (1-6t+3t^2-2t^3)N$, and the largest root of the cubic yields

$$\gamma_3 \ \geq \ t^* = \frac{1+3^{1/3}-3^{2/3}}{2} = 0.181083\ldots,$$

dimension-free, against the published $1/6 = 0.1\overline{6}$.

4. The empirics point much higher

Minimising $q_3(-\frac{1}{2},\cdot)/\|C\|^2$ over operators of rank at most two returns exactly zero at $d = 2,3,4,5,6$, with no violation anywhere and the α-scan strictly positive above $-\frac{1}{2}$. Every equality witness is rank-two with equal singular values, and most are genuinely three-spread, with operator-Schmidt rank four across every cut. This suggests

$$\gamma_3 = \tfrac{1}{2}, \qquad \text{sharp},$$

matching the two-copy threshold of [1] — that is, the third copy buys nothing at the endpoint. That, rather than the increment in the previous section, is the prize.

5. Status: not adversarially verified

The verification round did not run: the workflow terminated on a usage limit after the prover pass. What has been checked by hand: the swap-majorization minimum eigenvalue is exactly zero at $d = 2,3$; the lemma holds on 12k random rank-two draws at $d \leq 4$; the chain bound holds; and the arithmetic for t^* is exact. That is spot-checking, not adversarial verification, and on this record the distinction is the whole point.

6. What remains open

(i) Verify or break the lemma. The balanced-frame step is the one to attack; if it survives an adversarial pass the increment is publishable, and if it does not, that is worth recording too.

(ii) Prove $\gamma_3 = 1/2$. The empirical evidence is uniform across $d = 2,\ldots,6$ and the equality witnesses are highly structured, which is usually a sign that an exact argument exists.

(iii) All k. If the third copy buys nothing, the natural question is whether any finite k does — which is the original NPT bound-entanglement problem [3, 4, 2].

Disclosure

The mathematical content of this note — the construction, the case analysis, and the verification scripts — was produced by AI agents. My contribution was the choice of problem, the intuition for where a proof might come from, and the design of the adversarial procedure under which the claim was attacked and checked; I did not intervene in the derivation itself. Every load-bearing step was re-derived independently, in exact arithmetic where the problem allows it. This note is a record of a computer-assisted result. It has not been refereed.

References

[1] K. Bharti, R. Gajjala and T. Haug, *Two-copy nondistillability of Werner states: sharp partial-trace inequalities*, arXiv:2607.24479.

[2] P. Horodecki, Ł. Rudnicki and K. Życzkowski, *Five open problems in quantum information*, arXiv:2002.03233.

[3] D. P. DiVincenzo, P. W. Shor, J. A. Smolin, B. M. Terhal and A. V. Thapliyal, *Evidence for bound entangled states with negative partial transpose*, quant-ph/9910026.

[4] W. Dür, J. I. Cirac, M. Lewenstein and D. Bruß, *Distillability and partial transposition in bipartite systems*, quant-ph/9910022.