

Full-output classical simulation of lossy boson sampling at arbitrary mode density

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Abstract

We give a construction for sampling the full photon-count distribution of boson sampling with independent loss at any fixed rational transmission $0 < \eta < 1$. The proposed guarantee is uniform in the size parameters $m \geq n$: the total variation error, averaged over a Haar-random m -mode interferometer, is at most ε , with running time polynomial in $n, m, 1/\varepsilon$ and the binary input length. An exact particle-reduction identity bounds the high harmonic sectors without a dilute-mode approximation. We then combine input and output color coding with a collision cutoff whose retained heavy supports admit efficient enumeration. A two-projection coefficient identity evaluates the resulting consistent signed marginals in $\text{poly}(n, m) \exp(O(r))$ operations, where $r = O_\eta(1 + \log(1/\varepsilon))$. Finite matrix precision and finite random bits are included. The result is presented as a complete solution candidate with internal proof checks; independent expert review remains outstanding.

Exact algebraic checks and manuscript sources accompany this version. Their scope and the use of AI assistance are described in Section 6.

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1 Model, claim and relation to prior work

Boson sampling asks a classical computer to reproduce the photon-count statistics of a passive linear-optical experiment [1]. Photon loss suppresses many-particle interference, suggesting that only low orders need to be retained. The difficulty addressed here is to turn that truncation into a sampler when collisions are common, including when the number of modes equals the number of input photons.

Inject one indistinguishable photon into each of the first n inputs of an m -mode interferometer, with $m \geq n \geq 1$. Each photon survives independently with probability η . For $U \in U(m)$ and an occupation $s \in \mathbb{N}_0^m$, write $|s| = \sum_i s_i$ and $s! = \prod_i s_i!$. The target law is

$$p_{\eta,U}(s) = \sum_{\substack{I \subseteq [m] \\ |I|=|s|}} \eta^{|I|} (1-\eta)^{n-|I|} \frac{|\text{Per}(U_{s,I})|^2}{s!}. \quad (1.1)$$

Here $U_{s,I}$ repeats row i exactly s_i times and selects the columns in I . The empty permanent is one. In particular the output includes every collision pattern, together with the vacuum event. Total variation is $\text{TV}(p, q) = \frac{1}{2} \sum_s |p(s) - q(s)|$.

Theorem 1.1 (Main claim of this candidate). *Fix rational $0 < \eta < 1$. There is a randomized classical algorithm with the following property. For every $m \geq n \geq 1$ and rational accuracy $0 < \varepsilon < 1$, an entrywise encoding of U using $\text{poly}_\eta(n, m, \log(1/\varepsilon))$ bits suffices to produce an output law q_U satisfying*

$$\mathbb{E}_{U \sim \text{Haar}(U(m))} \text{TV}(q_U, p_{\eta,U}) \leq \varepsilon. \quad (1.2)$$

The contribution of matrix encoding is at most $\varepsilon/2$. The worst-case running time, including the generation of random bits, is polynomial in $n, m, 1/\varepsilon$ and the binary input length. The polynomial can depend on fixed η ; its exponent does not depend on ε .

The theorem addresses the question posed in the Quantum Information and Quantum Computation Open Problem Zoo [14]. The error is averaged over U , with random preprocessing included in q_U . The dependence on fixed η becomes expensive as η approaches one; Section 4.4 gives an explicit conservative bound. The special case $m = n = 1$ is a single Bernoulli draw; henceforth $m \geq 2$.

1.1 Prior work and the remaining mode-density restriction

Low-degree approximation under noise has a substantial history. Kalai and Kindler [2] studied Gaussian noise in permanent probabilities. Renema, Shchesnovich and García-Patrón [3] organized noisy boson-sampling probabilities by interference order, including the effect of photon loss. Shchesnovich [4] related Gaussian noise to a model combining loss and dark counts. Other efficient simulations use stronger loss [5] or depth-dependent attenuation in optical architectures [6]. These works provide distinct noise models and simulation regimes; the model here keeps a constant fraction of the input photons on average.

The closest full-output comparison is Go and Jeong [7]. Write N for their input photon count, k for the fixed survivor count and M for their mode count. Their Corollary 4 assumes $M \geq C_0 k^2 N$ and $M \geq C_B k^2 / (\varepsilon \delta)$, and gives TV error at most ε with Haar probability at least $1 - \delta$. Their Section XI identifies full-output simulation at comparable mode and photon counts as an open direction. Theorem 1.1 removes these mode restrictions. Its average-error formulation also gives a high-probability statement: running at mean tolerance $\varepsilon \delta$ bounds the probability of TV error exceeding ε by δ , with polynomial dependence on $1/\delta$. Thus the substantive scope difference is the allowed mode density. The additional birthday bound in [7] controls collisions, whose onset is studied in the bosonic birthday paradox [8].

Exact Haar analysis itself is established prior work. Nezami [9] uses representation theory to calculate moments of permanents of unitary submatrices. Arienzo et al. [10] provide the passive-bosonic operator decomposition and photon-number measurement frame used below. Mhiri et al. [11], Proposition 2 and Corollary 1, express Haar second moments through irreducible projection norms and derive anti-concentration results beyond the dilute regime. We use this sectorwise viewpoint to control loss, then construct the collision-resolving marginal evaluator needed for sampling.

1.2 Construction and notation

Trace-one particle reduction produces a factorial cancellation, after which independent loss suppresses harmonic degree l by the exact factor η^l . Collision patterns are grouped by their modes occupied at least twice, while singleton modes are color coded. Color coding originates with Alon, Yuster and Zwick [12]; its use with inverse determinants for this sampling problem follows the framework of [7]. The evaluator preserves both harmonic projections and has cost $\text{poly}(n, m) \exp(O(r))$. A signed-prefix construction [13, 7] then turns its compatible marginals into a probability sampler. The proposed extension is this combination of the loss estimate and the collision-resolving, two-projection evaluator at arbitrary mode density.

Let $H_k = \text{Sym}^k \mathbb{C}^m$, $D_k = \binom{m+k-1}{k}$ and $D_{-1} = 0$. Let $\Gamma_k(U)$ be the passive action on H_k . The trace-one j -particle reduction is $R_{k,j}$, and its Hilbert–Schmidt adjoint is $L_{k,j} = R_{k,j}^*$. Falling and rising factorials are written $(a)_l$ and $a^{\bar{l}}$. The input conditional on $K = k$ surviving photons is

$$\rho_{k,n} = \binom{n}{k}^{-1} \sum_{\substack{I \subseteq [n] \\ |I|=k}} |I\rangle\langle I|, \quad w_k = \Pr(K = k) = \binom{n}{k} \eta^k (1 - \eta)^{n-k}. \quad (1.3)$$

It obeys $R_{k,j} \rho_{k,n} = \rho_{j,n}$ and $\text{Tr} \rho_{j,n}^2 = \binom{n}{j}^{-1}$. We write $p_{U,k}$ for its output Fock diagonal. Thus (1.1) is the mixture $w_k p_{U,k}$ on the disjoint photon-number sectors.

2 An exact Haar harmonic error bound

2.1 Representation and measurement normalization

The conjugation representation on $\text{End}(H_k)$ is an orthogonal multiplicity-free sum of irreducible sectors $V_{k,l}$, $0 \leq l \leq k$, with highest weight $(l, 0, \dots, 0, -l)$ and dimension

$$d_l = D_l^2 - D_{l-1}^2. \quad (2.1)$$

Let $\Pi_{k,l}$ be the corresponding orthogonal projector. These representation facts are imported from [10], Lemma T1 and Proposition B2. The same source, Lemma T3 and Theorem T4, identifies the photon-number measurement frame. In our normalization its eigenvalue on sector l is

$$f_l = \frac{1}{D_l + D_{l-1}}, \quad f_0 = 1. \quad (2.2)$$

Indeed Fock dephasing $\mathcal{D}_k$ is averaging over the diagonal unitary torus. Its rank on $V_{k,l}$ is the zero-weight multiplicity $D_l - D_{l-1}$; dividing this trace by d_l gives (2.2). Equivalently, the source expression is

$$\frac{m-1}{(2l+m-1) \binom{l+m-2}{l}} = f_l.$$

For $X \in V_{k,l}$ this implies

$$\mathbb{E}_U \|\mathcal{D}_k(\Gamma_k(U) X \Gamma_k(U)^*)\|_{\text{HS}}^2 = f_l \|X\|_{\text{HS}}^2. \quad (2.3)$$

Dephasing commutes with sector projection, since the latter commutes with every conjugation action. Consequently diagonal vectors in different sectors remain orthogonal, even before Haar averaging.

2.2 Particle reduction

Lemma 2.1 (Reduction singular value). *Put*

$$B(k, l) = (l!)^2 \binom{m+k+l-1}{k-l}. \quad (2.4)$$

For $l \leq j \leq k$, the squared Hilbert–Schmidt singular value of $R_{k,j}$ on $V_{k,l}$ is

$$t_{k,j,l} = \left(\frac{(j)_l}{(k)_l} \right)^2 \frac{B(k, l)}{B(j, l)}. \quad (2.5)$$

Reduction annihilates sectors $l > j$.

Proof. Use $A_{k,l} = (a_1^\dagger)^l a_m^l$ on H_k . This is a highest-weight vector in $V_{k,l}$. The squared transition coefficient at occupation s is $(s_1 + l)!/s_1!$ times $s_m!/(s_m - l)!$. Its generating function in total photon number is

$$\frac{l!}{(1-z)^{l+1}} \frac{l!z^l}{(1-z)^{l+1}} \frac{1}{(1-z)^{m-2}} = \frac{(l!)^2 z^l}{(1-z)^{m+2l}}.$$

Coefficient extraction gives $\|A_{k,l}\|_{\text{HS}}^2 = B(k, l)$. The trace-one partial trace satisfies

$$R_{k,j}^*(A_{j,l}) = \frac{(j)_l}{(k)_l} A_{k,l}.$$

This follows by expressing the normally ordered l -particle observable as a sum over ordered distinct particle labels. Passing to a uniformly selected j -particle subsystem retains the displayed fraction of these labels. Equivariance and multiplicity-freeness make the map scalar between normalized highest weights. Their norms give (2.5). There is no copy of sector $l > j$ in the target. $\square$

For $W \in V_{l,l}$ define the compatible lift

$$J_{k,l}W = \frac{L_{k,l}W}{t_{k,l,l}} \quad (k \geq l). \quad (2.6)$$

It has squared norm $\|W\|_{\text{HS}}^2/t_{k,l,l}$ and l -particle reduction W . Transitivity of reduction and uniqueness in the irreducible sector give

$$R_{k,j}J_{k,l}W = J_{j,l}W \quad (j \geq l), \quad R_{k,j}J_{k,l}W = 0 \quad (j < l). \quad (2.7)$$

Lifts of diagonal operators are diagonal. Each $l > 0$ lift has trace zero.

2.3 Factorial cancellation and loss averaging

Lemma 2.2 (Dimension-uniform cancellation). *For $1 \leq l \leq k \leq n \leq m$,*

$$\frac{D_k f_l}{t_{k,l,l} \binom{n}{l}} = \frac{(k)_l}{(n)_l} \frac{(m+l)^{\bar{l}}}{(m+k)^{\bar{l}}} \frac{m+l-1}{m+2l-1} \leq \frac{(k)_l}{(n)_l}. \quad (2.8)$$

Proof. Substitute (2.5) and (2.2), then use

$$\frac{D_k}{\binom{m+k+l-1}{k-l}} = \frac{m^{\overline{l}}}{(k)_l(m+k)^{\overline{l}}}, \quad D_l + D_{l-1} = \frac{m^{\overline{l}}}{l!} \frac{m+2l-1}{m+l-1}.$$

Both remaining ratios in (2.8) are at most one. $\square$

Write $\rho_l^{(k)} = \Pi_{k,l} \rho_{k,n}$. Its reduction lies in the l th sector of $\rho_{l,n}$, so

$$t_{k,l,l} \|\rho_l^{(k)}\|_{\text{HS}}^2 \leq \binom{n}{l}^{-1}. \quad (2.9)$$

This is an upper bound, not an assertion that each sector saturates it. Define the normalized signed approximation

$$q_{U,k,r}(s) = \langle s | \Gamma_k(U) \sum_{l=0}^{\min(r,k)} \rho_l^{(k)} \Gamma_k(U)^* | s \rangle.$$

Equations (2.3)–(2.9) imply

$$\mathbb{E}_U D_k \sum_{|s|=k} |p_{U,k}(s) - q_{U,k,r}(s)|^2 \leq \sum_{l=r+1}^k \frac{(k)_l}{(n)_l}. \quad (2.10)$$

To pass to loss, use the probability reference w_k/D_k in Cauchy–Schwarz and then Jensen’s inequality. Since $\mathbb{E}(K)_l/(n)_l = \eta^l$ exactly,

$$\begin{aligned} \mathbb{E}_U \|p_{\eta,U} - q_{U,r}\|_1 &\leq \left(\sum_k w_k D_k \mathbb{E}_U \sum_{|s|=k} |p_{U,k}(s) - q_{U,k,r}(s)|^2 \right)^{1/2} \\ &\leq \left(\sum_{l=r+1}^n \eta^l \right)^{1/2} \leq \frac{\eta^{(r+1)/2}}{\sqrt{1-\eta}}, \end{aligned} \quad (2.11)$$

where $q_{U,r}(s) = w_{|s|} q_{U,|s|,r}(s)$. The tail is zero for $r = n$. This argument includes $K = n$ and every colliding output; it does not remove rare survivor counts or replace Haar matrices by Gaussian matrices.

3 A collision cutoff and one consistent random signed law

3.1 Counting the heavy supports

For an l -photon occupation v , define its deficit $d(v) = l - |\text{supp}(v)|$ and heavy support $H(v) = \{i : v_i \geq 2\}$. Write $b_i = v_i$ on $H(v)$ and $s = l - \sum_{i \in H(v)} b_i$. Thus s singleton modes remain and $d(v) = \sum_{i \in H(v)} (b_i - 1)$.

The uniform law on the D_l occupations has a Polya-urn realization: start with one ball of each of m colors and make l additions. After a additions, the probability of selecting an occupied color is at most $2a/m$. Every such repeat increases the deficit by one. A union bound over D selected addition times, using $\binom{l}{D} \leq (el/D)^D$, gives

$$\Pr_{v \text{ uniform}} [d(v) \geq D] \leq \min \left\{ 1, \left(\frac{6l^2}{mD} \right)^D \right\} \quad (1 \leq D < l). \quad (3.1)$$

For $D = l$ the probability is zero. Given $0 < \tau < 1$, let $D(l)$ be the first $D \in \{1, \dots, l\}$ for which this upper bound, with the $D = l$ value set to zero, is at most τ . Retain exactly the occupations with $d(v) < D(l)$.

Lemma 3.1 (Efficient heavy-support enumeration). *With $C = 4$,*

$$\sum_{q=0}^{\min(D(l)-1, \lfloor l/2 \rfloor)} \binom{m}{q} \leq (l+1)\tau^{-1}e^{Cl}. \quad (3.2)$$

For each support, its possible heavy occupancies can be enumerated in $\text{poly}(l)2^l$ time.

Proof. For $D(l) = 1$ only the empty support occurs. Otherwise $d = D(l) - 1$ failed the cutoff test, so $d \log(md/(6l^2)) < \log(1/\tau)$. For $1 \leq q \leq d \leq l \leq m$, monotonicity of $x \log(em/x)$ gives

$$\begin{aligned} \log \binom{m}{q} &\leq q \log(em/q) \leq d \log(em/d) \\ &< \log(1/\tau) + d \log(6e) + 2d \log(l/d) \leq \log(1/\tau) + Cl. \end{aligned}$$

Here $d \leq l$ and $d \log(l/d) \leq l/e$, so $C = \log(6e) + 2/e < 4$ suffices. Sum over at most $l+1$ support sizes. For a support of size q , recursively enumerate $b_i \geq 2$ with $\sum_i b_i \leq l$, restricting the next value so that the remaining positions can each receive at least two. The number of assignments is bounded by the number of compositions of totals at most l , hence by 2^l ; the tree overhead is polynomial in l . Apply the deficit test to these assignments. $\square$

The cutoff concerns occupations of the low harmonic degree l , not the physical survivor number k . It does not declare all physical output collisions rare or replace them by a common symbol.

3.2 Input and output colors

Following the color-coding method [12], for each $1 \leq l \leq r$ choose independently a uniform input coloring $h_l : [n] \rightarrow [l]$. Set $p_l = l!/l^l$ and define

$$X_l = \frac{1}{p_l \binom{n}{l}} \sum_{\substack{I \subseteq [n], |I|=l \\ h_l(I)=[l]}} |I\rangle \langle I|, \quad W_l = \Pi_{l,l} X_l. \quad (3.3)$$

The trace of X_l can fluctuate. Its expectation is $\rho_{l,n}$, and orthogonality of the diagonal terms gives

$$\mathbb{E} \|W_l\|_{\text{HS}}^2 \leq \mathbb{E} \|X_l\|_{\text{HS}}^2 = \frac{1}{p_l \binom{n}{l}}. \quad (3.4)$$

Every W_l is traceless.

For each $s \in \{1, \dots, l\}$ choose a uniform output coloring $g_{l,s} : [m] \rightarrow [s]$. Let $S(v)$ be the singleton support of v and $s = |S(v)|$. Define $p_0 = 1$ and

$$a_l(v) = \mathbf{1}_{d(v) < D(l)} \frac{\mathbf{1}_{g_{l,s}(S(v))=[s]}}{p_s}. \quad (3.5)$$

The empty singleton condition is automatically satisfied. Let $\mathcal{D}_{a_l}$ discard off-diagonal entries and multiply the diagonal at v by $a_l(v)$. Its mean is the projection $\mathcal{D}_{\text{good}}$ onto retained diagonal labels, and $\mathbb{E}_g a_l(v)^2 \leq e^l$.

For one collection of all colors set

$$A_l(U) = \Pi_{l,l} \mathcal{D}_{a_l} (\Gamma_l(U) W_l \Gamma_l(U)^*), \quad B_k(U) = \frac{I}{D_k} + \sum_{l=1}^{\min(r,k)} J_{k,l} A_l(U). \quad (3.6)$$

The outer projection places A_l in $V_{l,l}$, so (2.7) applies. The matrices B_k are diagonal, Hermitian, trace one and particle compatible. Average T independent complete color collections, keep them fixed throughout a sample, and denote the resulting matrices by $\overline{B}_k$. Then

$$\tilde{q}_U(s) = w_{|s|} \langle s | \overline{B}_{|s|}(U) | s \rangle$$

is a single normalized signed law. Reusing the same colors for each prefix preserves marginal consistency.

3.3 Bias, variance and the total error

For a fixed $W \in V_{l,l}$, Schur orthogonality yields

$$\mathbb{E}_U |\langle v | \Gamma_l(U) W \Gamma_l(U)^* | v \rangle|^2 = \frac{\|W\|_{\text{HS}}^2 \|\Pi_{l,l}(|v\rangle\langle v|)\|_{\text{HS}}^2}{d_l} \leq \frac{\|W\|_{\text{HS}}^2}{d_l}. \quad (3.7)$$

At most τD_l labels are discarded. Moreover,

$$\frac{D_l}{d_l} = \frac{D_l}{D_l - D_{l-1}} f_l = \frac{m + l - 1}{m - 1} f_l \leq 3f_l,$$

because $m \geq l$ and $m \geq 2$. In the mean of (3.6), use the actual input $W = \Pi_{l,l} \rho_{l,n}$, whose squared norm is at most $\binom{n}{l}^{-1}$. Projection contraction, the lift norm, and (2.8) bound the D_k -weighted squared bias at degree l by

$$3\tau \frac{(k)_l}{(n)_l}. \quad (3.8)$$

Here the unmasked target is the exact dephased harmonic component: the equivariant lift commutes with conjugation and with torus dephasing.

For the variance, condition on X_l and U . Equation (3.5) and projection contraction imply

$$\mathbb{E}_g \|A_l(U)\|_{\text{HS}}^2 \leq e^l \|\mathcal{D}_l(\Gamma_l(U) W_l \Gamma_l(U)^*)\|_{\text{HS}}^2.$$

Average over U and then h_l . Since $p_l^{-1} \leq e^l$, equations (2.3) and (3.4) give $\mathbb{E} \|A_l(U)\|_{\text{HS}}^2 \leq e^{2l} f_l / \binom{n}{l}$. Centered variance is no larger. Independence of the T collections and (2.8) therefore bound the weighted lifted variance by

$$T^{-1} e^{2l} \frac{(k)_l}{(n)_l}. \quad (3.9)$$

Errors at different degrees are orthogonal in diagonal Hilbert–Schmidt space. Applying the same weighted Cauchy–Schwarz argument as in (2.11), separately to bias and centered fluctuation, gives

$$\mathbb{E}_{U, \text{colors}} \|p_{\eta, U} - \tilde{q}_U\|_1 \leq \frac{\eta^{(r+1)/2}}{\sqrt{1-\eta}} + \sqrt{\frac{3\tau}{1-\eta}} + \left(T^{-1} \sum_{l=1}^r e^{2l} \eta^l \right)^{1/2}. \quad (3.10)$$

Omit the first term when $r = n$. Correlations among the output-color indicators are included: the argument does not assume independent monomials.

Select r as the minimum of n and the least nonnegative integer for which $\eta^r \leq \varepsilon^2(1-\eta)/64$, and put

$$\tau = \frac{\varepsilon^2(1-\eta)}{192}, \quad T = \left\lceil 64\varepsilon^{-2} r 9^r \right\rceil. \quad (3.11)$$

Here $r \geq 1$. Each term in (3.10) is at most $\varepsilon/8$, hence the total is at most $3\varepsilon/8$. Parameter selection uses rational comparisons. For fixed η , $r = O_\eta(1 + \log(1/\varepsilon))$, while T and τ^{-1} are polynomial in $1/\varepsilon$.

4 Evaluating prefixes while retaining collisions

4.1 Both projections in reduced-particle coordinates

On H_l , compute rational coefficients c_a by

$$c_l = 1, \quad c_j = -\frac{\sum_{a=j+1}^l c_a t_{l,a,j}}{t_{l,j,j}} \quad (j = l-1, \dots, 0). \quad (4.1)$$

The action on each irreducible sector gives the exact identity

$$\Pi_{l,l} = \sum_{a=0}^l c_a L_{l,a} R_{l,a}. \quad (4.2)$$

These equations invert operator maps; they do not divide by a possibly vanishing state component.

Fix l , suppress the index on its input coloring, and write $N_c = |h_l^{-1}(c)|$. For a colorful a -subset I ,

$$\langle I | R_{l,a} X_l | I \rangle = \frac{\prod_{c \notin h_l(I)} N_c}{p_l \binom{n}{l} \binom{l}{a}}. \quad (4.3)$$

All other diagonal entries vanish. This simply counts colorful l -subsets extending I . The lift of a diagonal matrix satisfies the hypergeometric formula

$$\langle v | L_{l,a}(Z) | v \rangle = \binom{l}{a}^{-1} \sum_{\substack{u \leq v \\ |u|=a}} \binom{v}{u} \langle u | Z | u \rangle, \quad \binom{v}{u} = \prod_i \binom{v_i}{u_i}. \quad (4.4)$$

Equivariance, (4.3) and (4.4) give

$$\begin{aligned} Z_l(v) &:= \langle v | \Gamma_l(U) \Pi_{l,l} X_l \Gamma_l(U)^* | v \rangle \\ &= \sum_{a=0}^l \frac{c_a}{p_l \binom{n}{l} \binom{l}{a}^2} \sum_{\substack{I \subseteq [n], |I|=a \\ I \text{ colorful}}} \left(\prod_{c \notin h_l(I)} N_c \right) \sum_{\substack{u \leq v \\ |u|=a}} \binom{v}{u} \frac{|\text{Per}(U_{u,I})|^2}{u!}. \end{aligned} \quad (4.5)$$

One factor $\binom{l}{a}$ comes from input reduction and the other from lifting.

Represent a k -photon outcome by a uniformly shuffled ordered list of its photons. A particular prefix of length ℓ has occupation h , with $\sum_i h_i = \ell$. Its probability under a compatible family is $h!/\ell!$ times the h th Fock diagonal of the ℓ -particle matrix. Degrees $l > \ell$ contribute zero. For $l \leq \ell$, moving the outer projector in (3.6) onto the prefix test gives

$$\begin{aligned} F_{b,h}(x) &= \sum_{t=0}^{\min(b,h)} \binom{b}{t} \binom{h}{t} x^t, \\ K_{\ell,l}(h, v) &= \sum_{j=0}^l \frac{c_j}{\binom{l}{j} \binom{\ell}{j}} [x^j] \prod_i F_{v_i, h_i}(x). \end{aligned} \quad (4.6)$$

To see this, apply (4.2) to $R_{\ell,l} |h\rangle \langle h|$, use reduction transitivity, and then (4.4). The convolution of the two hypergeometric weights is exactly the coefficient in (4.6). Thus the degree- l contribution to the ordered prefix is

$$\frac{h!}{\ell! t_{\ell,l,l}} \sum_{|v|=\ell} a_l(v) Z_l(v) K_{\ell,l}(h, v). \quad (4.7)$$

The scalar reference contributes $h!/(\ell! D_\ell)$. Summing degrees and averaging the fixed T collections gives the required prefix marginal.

4.2 A finite-ring coefficient identity

Fix an enumerated heavy support H and its occupancies $b_i \geq 2$. Put $s = l - \sum_{i \in H} b_i$. Introduce commuting square-zero variables $X_1, \dots, X_s$ and $Y_1, \dots, Y_l$, heavy variables z_i truncated above degree b_i , and x truncated above degree l . These operations take place in a finite commutative quotient ring over Gaussian rationals.

Let V be the first n columns of U , $Y = \text{diag}(Y_{h_l(1)}, \dots, Y_{h_l(n)})$, and define

$$f_i(x) = 1 + h_i x, \quad Z_i = \begin{cases} X_{g_{l,s}(i)} f_i(x), & i \notin H, \ s > 0, \\ 0, & i \notin H, \ s = 0, \\ z_i, & i \in H. \end{cases}$$

Compute

$$G = \left[\prod_{i \notin H} (1 + X_{g_{l,s}(i)} f_i(x)) \right] \det(I_m - \text{diag}(Z) V Y V^*)^{-1} \prod_{i \in H} F_{b_i, h_i}(x), \quad (4.8)$$

where the first product is one when $s = 0$. It can be grouped by color since each $X_c^2 = 0$.

For each $T_0 \subseteq [l]$, $0 \leq u_i \leq b_i$ on H , and $0 \leq j \leq l$, extract the coefficient of

$$X_1 \cdots X_s \prod_{c \in T_0} Y_c \prod_{i \in H} z_i^{u_i} x^j.$$

Multiply it by

$$\frac{c_{|T_0|} c_j \prod_{c \notin T_0} N_c \prod_{i \in H} \binom{b_i}{u_i}}{p_l \binom{n}{l} \binom{l}{|T_0|}^2 \binom{l}{j} \binom{\ell}{j} p_s}. \quad (4.9)$$

Sum over coefficients and all retained heavy patterns, then multiply by $h! / (\ell! t_{\ell, l, l})$. The result is (4.7).

Proof of the coefficient identity. MacMahon's inverse-determinant identity reads

$$\det(I - \text{diag}(Z) C)^{-1} = \sum_u \frac{\text{Per}(C_u)}{u!} Z^u,$$

where C_u repeats both rows and columns. Put $C = V Y V^*$ and expand by permanent Cauchy–Binet. A squarefree color coefficient selects distinct input modes with precisely those colors; its value is the sum of $|\text{Per}(V_{u, I})|^2 / u!$ over those modes.

A singleton mode selected actively in the determinant contributes one X and one $f_i(x)$. If it is inactive in u , the external product in (4.8) can instead select it as a spectator in v . Selecting the same color twice vanishes, so an active singleton cannot also be counted as a spectator, and distinct singleton modes cannot share a retained color. Consequently extraction of $X_1 \cdots X_s$ selects exactly the colorful singleton supports, with one $f_i(x)$ per selected mode. Heavy modes are fixed by H, b and may participate u_i times; their binomial weights are inserted by (4.9). The last product in (4.8) supplies their factors in (4.6). The remaining weights are precisely (4.5) and (4.6). This includes empty input-color classes and $s = 0$. $\square$

4.3 Arithmetic complexity

The ring has at most

$$(l+1) 2^{s+l} \prod_{i \in H} (b_i + 1) \leq (l+1) 4^l \quad (4.10)$$

coefficients, because $b + 1 \leq 2^b$ and $s + \sum_i b_i = l$. Schoolbook ring multiplication costs $\exp(O(l))$ scalar operations. Sylvester's determinant identity reduces (4.8) to $\det(I_n - M)^{-1}$ with $M = YV^* \text{diag}(Z)V$. Every entry of M has positive Y -degree, so $M^{l+1} = 0$ in this ring. Evaluate

$$\det(I_n - M)^{-1} = \exp\left(\sum_{a=1}^l \frac{\text{Tr}(M^a)}{a}\right), \quad (4.11)$$

truncating the exponential at degree l . This takes $\text{poly}(n, m, l) \exp(O(l))$ operations, with neither an $n!$ determinant expansion nor enumeration of n^l input subsets.

Lemma 3.1, all degrees, T collections, and at most nm prefix queries yield the arithmetic bound

$$\text{poly}(n, m, r) T \tau^{-1} \exp(O(r)). \quad (4.12)$$

Since $r = O_\eta(1 + \log(1/\varepsilon))$, this is polynomial in the required size parameters for each fixed η . Section 5 supplies the bit complexity and the sampler.

4.4 Dependence on transmission and accuracy

The dependence on accuracy can be made explicit, although the following upper bound is deliberately conservative. Put $a_\eta = \log(1/\eta)$. The choice in (3.11) satisfies

$$r \leq \left\lceil \frac{\log(64/((1-\eta)\varepsilon^2))}{a_\eta} \right\rceil.$$

The number of color collections alone is therefore

$$T = \tilde{O}_\eta(\varepsilon^{-\beta(\eta)}), \quad \beta(\eta) = 2 + \frac{2 \log 9}{a_\eta}.$$

This exponent omits the work within each collection. To bound that work, use $C = 4$ in Lemma 3.1. The heavy occupancy assignments contribute 2^l , and schoolbook multiplication in the ring of (4.10) contributes at most $\text{poly}(l)16^l$. Thus (4.12) can be bounded explicitly by

$$\text{poly}(n, m, r) T \tau^{-1} (32e^4)^r \leq \text{poly}(n, m, r) O_\eta(\varepsilon^{-4}(288e^4)^r). \quad (4.13)$$

For fixed η , a conservative arithmetic-operation count is consequently

$$\text{poly}(n, m) \tilde{O}_\eta(\varepsilon^{-\alpha(\eta)}), \quad \alpha(\eta) = 4 + \frac{2 \log(288e^4)}{a_\eta}. \quad (4.14)$$

The tilde hides powers of $\log(1/\varepsilon)$. These are upper bounds for the specified construction, not optimized exponents or measured runtimes.

Transmission η	Color count $\beta(\eta)$	Full arithmetic bound $\alpha(\eta)$
0.25	5.17	17.94
0.50	8.34	31.88
0.75	17.28	71.18
0.90	43.71	187.43

In particular, a small exponent for the number of color collections does not give the exponent of the whole algorithm. These bounds establish polynomial scaling for fixed transmission, while leaving practical efficiency unresolved.

5 A proper sampler with polynomial bit complexity

5.1 Sampling a normalized signed law

Sampling from approximately correct signed marginals appears in Bremner, Montanaro and Shepherd [13], Section 3.2 and Appendix B, for binary strings, and in [7], Lemma 8. Here the signed law has mass exactly one. We prove the resulting finite-alphabet bound and its dependence on marginal precision.

Lemma 5.1 (Clipped-prefix sampling). *Let q be a real signed function of total mass one on length- k strings, and suppose all prefix marginals can be queried. At a prefix v , choose child w proportionally to $q(w)_+ = \max(q(w), 0)$. This defines a probability law S and, for every probability law p ,*

$$S(v) \leq q(v)_+ \quad \text{at every prefix}, \quad \text{TV}(S, p) \leq \frac{1}{2} \|q - p\|_1. \quad (5.1)$$

If each queried child marginal is replaced by a value within δ of its true value, the change in the sampled law is at most $2km^k\delta$ for an alphabet of size m . A fallback distribution may be used if all approximate positive weights vanish.

Proof. The empty-prefix marginal equals one. Write $C(v) = \sum_{w \text{ child of } v} q(w)_+ \geq q(v)$. Induction gives $S(v) \leq q(v)_+$ and, at any reachable prefix, $q(v) > 0$ and $C(v) > 0$. Then $S(w) = S(v)q(w)_+/C(v) \leq q(w)_+$. At leaves,

$$\text{TV}(S, p) = \sum_x (S(x) - p(x))_+ \leq \sum_x (q(x) - p(x))_+ = \frac{1}{2} \|q - p\|_1.$$

Clipping changes each child weight by at most δ . The conditional distributions before and after approximation differ in TV by at most $2m\delta/C(v)$, capped at one. If all approximate weights vanish, $C(v) \leq m\delta$, so the same capped estimate covers an arbitrary fallback. Couple paths until their first discrepancy. Weighting a node by the probability $S(v) \leq C(v)$ bounds its contribution by $2m\delta$. At most km^{k-1} nonterminal prefixes exist, which proves the stated conservative bound. The sampler visits only one path and makes at most km oracle queries; the full tree is used only in the error estimate. $\square$

Apply the lemma conditional on $K = k$ to the ordered law of $\bar{B}_k$, with all colors fixed. Forgetting order cannot increase TV. Averaging over K , colors and U , convexity and (3.10) give an ideal-sampling error at most $3\varepsilon/8$, even using the weaker bound $\text{TV} \leq \|q - p\|_1$. This conservative allocation leaves room for encoding and random-bit approximation.

5.2 Uniform matrix precision

The prefix identities in Section 4 are polynomials in entries of U and their conjugates of degree at most $2r$. They remain explicit polynomials when evaluated at a rounded matrix, even though that matrix need not be unitary. We bound their perturbation directly; no Haar theorem for rounded matrices is assumed.

One deliberately loose coefficient bound is as follows:

$$\begin{aligned} M_0 &= 4^r(m + 2r)^r, & C_0 &= (1 + rM_0)^r, \\ B_0 &= 1 + (r + 1)^6 C_0^2 (r!)^2 [64(m + 2r)(n + 1)^6]^r. \end{aligned} \quad (5.2)$$

Every top-projector coefficient for $l \leq r$ satisfies $|c_j| \leq C_0$. Indeed $t_{l,a,j} \leq (m + 2r)^r$ and $t_{l,j,j} \geq \binom{l}{j}^{-2} \geq 4^{-r}$, and backward substitution in (4.1) gives this bound.

For each fixed color collection and ordered prefix, the sum of absolute polynomial coefficients is at most B_0 . A direct bound from (4.5)–(4.7) counts at most $(m + r)^r$ output occupations,

$(n+1)^r$ input subsets and 2^r suboccupations. Bound $\binom{v}{u} \leq 2^r$, the squared-permanent coefficient sum by $(r!)^2$, unused-color counts by $(n+1)^r$, and both color reweightings by $e^{2r} < 9^r$. The projector coefficients contribute C_0^2 . Also

$$\prod_i F_{v_i, h_i}(1) = \prod_i \binom{v_i + h_i}{v_i} \leq \prod_i (h_i + 1)^{v_i} \leq (n+1)^r.$$

The inverse lift factor is at most n^{2r} , since $t_{\ell, l, l} \geq \binom{\ell}{l}^{-2}$. Remaining binomial denominators only decrease the bound, and $h!/l! \leq 1$. The sums over a, j, l and the reference term are absorbed in (5.2). Averaging the T collections does not increase the bound.

If both matrices have entries of modulus at most two, telescoping a degree- $2r$ monomial bounds each prefix change by

$$B_0(2r)2^{2r} \max_{i,j} |U_{ij} - V_{ij}|. \quad (5.3)$$

Round real and imaginary entries to b_{entry} fractional bits. Set

$$\delta = \frac{\varepsilon}{8nm^n}, \quad 2^{b_{\text{entry}}} \geq \frac{4r2^{2r}B_0}{\delta}. \quad (5.4)$$

The complex entry error is at most $2 \cdot 2^{-b_{\text{entry}}}$, so every prefix error is at most δ , uniformly in U and the colors. The required b_{entry} is polynomial in $n, m, \log(1/\varepsilon)$ for fixed η . By Lemma 5.1, the total encoding contribution is at most $2nm^n\delta = \varepsilon/4$, which is below the original $\varepsilon/2$ allowance.

5.3 Exact arithmetic at the encoded input

Evaluate the prefix polynomial at the encoded dyadic matrix with exact Gaussian rational arithmetic. This removes a separate numerical-arithmetic error budget. Factorials and binomials have polynomial bit length. The rational entries of the triangular system (4.1) have $O(r \log(m+r))$ bits. Clearing at most $O(r^2)$ denominators and bounding the resulting determinants gives polynomial bit length for all c_j .

In (4.11), Y -degree is at most r , and matrix-entry degree is at most $2r$. Products of the finite-series factorial denominators have polynomial logarithmic size. A common dyadic input denominator has exponent at most $2rb_{\text{entry}}$. The remaining reduction, coloring, lift and averaging denominators have polynomial bit length; taking their product over the at most polynomially many distinct parameter indices is a sufficient common denominator. Matrix-index summation and the bounded-degree products add only polynomially many magnitude bits. Thus intermediate numerators and denominators have polynomial bit length, despite the $\exp(O(r))$ coefficient count. Exact arithmetic preserves the polynomial running time in (4.12).

5.4 Finite random bits and the final budget

An exact draw from a non-dyadic rational probability does not in general admit a fixed finite number of fair random bits. Replace all such draws explicitly. For a categorical law on $a \leq m$ outcomes, round cumulative thresholds down to the grid $2^{-b_{\text{coin}}}$ and use a uniform b_{coin} -bit integer. The TV discrepancy is at most $a2^{-b_{\text{coin}}} \leq m2^{-b_{\text{coin}}}$. Threshold comparisons are exact.

One sample uses at most

$$L = T \left(nr + \frac{mr(r+1)}{2} \right) + 2n \quad (5.5)$$

categorical choices: all input and output colors, n Bernoulli loss decisions whose sum is K , and at most n output-mode draws. Choose

$$2^{b_{\text{coin}}} \geq 8mL/\varepsilon. \quad (5.6)$$

A union coupling bounds their total contribution, including the change in the color distribution, by $\varepsilon/8$. Both L and the arithmetic work are polynomial, and $b_{\text{coin}} = O(\log(Lm/\varepsilon))$. There is no unbounded rejection loop. For $m = n = 1$, use the same threshold construction with the two Bernoulli outcomes directly.

Completion of Theorem 1.1. Use (3.11), the finite-ring oracle, (5.4), and the finite-bit choices above. Sample K , use clipped-prefix sampling conditional on it, and return the output occupation. The preceding bounds give

$$\mathbb{E}_U \text{TV}(q_U, p_{\eta,U}) \leq \frac{3\varepsilon}{8} + \frac{\varepsilon}{4} + \frac{\varepsilon}{8} = \frac{3\varepsilon}{4} < \varepsilon.$$

Equation (4.12) and the bit bounds prove worst-case polynomial time for fixed η . The input encoding meets the required bit count and error allocation. $\square$

For one sample, compute $r, \tau, T, D(l)$ and the projector coefficients; draw and retain the color collections; sum n finite-bit Bernoulli decisions to obtain K ; at each ordered prefix, evaluate its m children using (4.8)–(4.9), clip and normalize, and draw the next mode. Return the occupation of the resulting list. A zero approximate normalization uses any fixed fallback. This summary specifies the mathematical construction; the accompanying small-case checkers are validation tools, not an optimized implementation of this full algorithm.

6 Verification, reproducibility and scope of the claim

The manuscript combines the exact harmonic estimate, compatible particle marginals, collision-stratified evaluator and finite-bit clarification. The scientific scope is the same in the abstract and Theorem 1.1: fixed rational transmission, every $m \geq n$, complete Fock outcomes, a Haar-average TV guarantee and polynomial bit complexity. The proof does not address partial distinguishability, nonuniform losses, arbitrary input states or a limit in which η approaches one with the size parameters.

The accompanying exact checker uses rational real and imaginary parts. It compares a Leibniz determinant followed by a finite geometric-series inverse to separately enumerated permanent expressions. It imports no earlier candidate checker. Its determinant implementation differs from the trace-power exponential used by the earlier numerical control. The recorded controls are:

- 329 exact frame/factorial identities and inequalities over small dimensions;
- eight coefficient cases at $l = 2, 3, 4$, including spectators, full collisions, two heavy modes and an empty input-color class; seven cases are nonzero;
- mutation controls detecting omission of spectators in four cases, of the second binomial denominator in seven, and of heavy-mode binomial weights in five.

The acceptance comparisons use no floating tolerance. The public runner checks the package hashes and repeats these finite algebraic checks. These controls probe fragile normalization and multiplicity factors; they do not replace the analytic proof for all dimensions, certify its asymptotic complexity experimentally, or establish independent expert agreement.

The source comparison distinguishes the imported representation and measurement-frame formulas [10], related Haar-moment methods [9, 11], and the preceding simulation algorithms

from the loss estimate and collision-resolving evaluator developed here. The comparison with [7] establishes a difference in stated mode assumptions. It does not settle priority relative to all concurrent work.

This manuscript was developed with AI assistance under human direction, including derivation, literature comparison, computation, internal review and drafting. The exact checker and analytic audit belong to that same research workflow. Independent expert review remains outstanding. The website release presents a complete solution candidate; the problem catalog’s status is maintained separately.

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