

Universality from Every Fixed Non-Gaussian Polynomial Generator

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Abstract

Let H_* be any real Weyl-ordered polynomial of degree greater than two, essentially self-adjoint on Schwartz space in any finite number of bosonic modes. We prove that Gaussian unitaries together with e^{-itH_*} , for arbitrary real t , generate a strongly dense subgroup of the full unitary group. The same finite products approximate any target unitary uniformly on input states with bounded mean photon number, including arbitrary reference systems. A squeezing limit extracts a pure position power; finite differences and a justified product formula produce cubic and quartic phases. A diagonal quartic drift then separates Fock transitions. Analytic-vector estimates justify its unbounded spectral filters, whose limits are bounded single-edge Hamiltonians. This gives a candidate affirmative answer to QIQCOP problem `op_64046727b81b4024`, with no Schwartz-preservation assumption on the target. No efficiency or intermediate-energy bound is claimed.

1 Control model and statement

On $\mathcal{H} = L^2(\mathbb{R}^n)$, use canonical coordinates with $[q_j, p_l] = i\delta_{jl}$ on $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$. Let $N_j = (q_j^2 + p_j^2 - 1)/2$ and $N = \sum_j N_j$. All Gaussian unitaries, including metaplectic transformations and displacements, are available. Fix a real Weyl polynomial H_* of total degree $d > 2$, essentially self-adjoint on $\mathcal{S}$. The other allowed operations are e^{-itH_*} for every $t \in \mathbb{R}$. An allowed circuit is a finite product of these operations; there are no extra resource modes.

Arzani, Booth, and Chabaud prove approximation results using *target-dependent* polynomial Hamiltonians and distinguish the fixed non-Gaussian generator question in their Discussion [2]. Keyl's related infinite-dimensional control theorem uses bounded controls [5]. Wu, Tarn, and Li study smooth state-orbit controllability and Kerr examples [6]. The argument below addresses the different quantifiers and the unbounded operators in the fixed-generator catalog problem [1] directly.

For a unitary U , write $\mathcal{U}(\rho) = U\rho U^*$. Define the reference-assisted input-energy distance

$$d_E(U, V) = \sup_{R, \rho_{AR}: \text{Tr}(\rho_A N) \leq E} \|[(\mathcal{U} - \mathcal{V}) \otimes \text{id}_R](\rho_{AR})\|_1, \quad (1.1)$$

where the supremum is over states and arbitrary reference systems.

Theorem 1. *For every finite $n \geq 1$ and every fixed H_* as above, the allowed finite products are strongly dense in the unitary group of $\mathcal{H}$. Consequently, for every unitary U , every finite $E \geq 0$, and every $\varepsilon > 0$, an allowed finite product V satisfies $d_E(U, V) < \varepsilon$.*

The catalog asks this for targets preserving Schwartz space. The theorem does not need that additional condition. Essential self-adjointness of the *given* polynomial and access to both signs of its evolution time remain hypotheses throughout.

2 Operator approximation facts

Let G be the strong closure of allowed finite products, taken relative to the unitary group. Multiplication is strongly continuous on unitaries. If $U_m \rightarrow U$ strongly with U unitary, then $\|(U_m^* - U^*)\psi\| = \|\psi - U_m U^* \psi\| \rightarrow 0$. Thus G is a group. All its elements are bounded unitaries, even when the generators used to describe them are unbounded.

Lemma 2. *Suppose A_m, A are self-adjoint, D is a core for A contained in every $\text{Dom}(A_m)$, and $A_m \psi \rightarrow A \psi$ for $\psi \in D$. Then A_m tends to A in strong resolvent sense and $e^{itA_m} \rightarrow e^{itA}$ strongly for every real t .*

Proof. For $\psi \in D$,

$$(A_m - i)^{-1}(A - i)\psi - \psi = (A_m - i)^{-1}(A - A_m)\psi \rightarrow 0.$$

The range $(A - i)D$ is dense and the resolvent norms are at most one; this proves strong resolvent convergence. The strong functional-calculus conclusion holds for exponentials as well. One way to see this last step is to first approximate by continuous functions of compact support and then bound the tails: spectral measures for each fixed vector have fixed total mass, and the limiting self-adjoint operator has that same mass, so continuous cutoffs make both tails arbitrarily small. This gives the claim for the bounded continuous function e^{itx} . $\square$

Lemma 3. *Let $B_1, \dots, B_r$ be self-adjoint. Suppose D lies in all their domains and $\sum_l B_l$ on D is essentially self-adjoint with closure A . Then for each real t ,*

$$\left(e^{itB_1/m} \dots e^{itB_r/m}\right)^m \rightarrow e^{itA} \quad \text{strongly.} \quad (2.1)$$

Proof. Set $T(s) = e^{isB_1} \dots e^{isB_r}$ for $s \geq 0$. This strongly continuous contraction family satisfies $T(0) = I$. Telescoping the product at zero gives $T'(0)\psi = iA\psi$ on D ; no invariance of D under the individual factors is required. For clarity, the full strong derivative, wherever defined, is dissipative: differentiating $\|T(s)\psi\| \leq \|\psi\|$ at zero gives the requisite real-part inequality. Since it extends the essentially skew-adjoint restriction $iA|_D$, its closure is exactly the maximal dissipative operator iA . Indeed, no dissipative extension of a maximal dissipative operator is proper. Chernoff's contraction product formula [3, Proposition 1.1] therefore applies. Negative t follows by replacing all B_l with $-B_l$. Real coefficients may be absorbed into the operators. $\square$

3 Extracting a position power

Let h_d be the nonzero homogeneous degree- d component of the real Weyl symbol of H_* . Choose a nonzero phase-space vector v with $h_d(v) \neq 0$. Extend v to a symplectic basis and apply the corresponding metaplectic change of coordinates. The resulting polynomial K has q_1^d coefficient $c = h_d(v) \neq 0$. Its self-adjoint realization is the unitary conjugate of $\overline{H_*}$, with Schwartz as a core.

For $s > 0$, apply a Gaussian squeeze with $q_1 \mapsto sq_1$, $p_1 \mapsto s^{-1}p_1$, leaving other coordinates fixed, and multiply the conjugate by s^{-d} . Denote the resulting self-adjoint operator by A_s . On

Schwartz, metaplectic covariance of Weyl quantization gives the indicated polynomial substitution. A monomial with powers a, b of q_1, p_1 has squeeze weight $a - b$. Among monomials of total degree at most d , only q_1^d can have weight d . Hence

$$A_s \psi \longrightarrow c q_1^d \psi \quad (\psi \in \mathcal{S}, s \rightarrow \infty). \quad (3.1)$$

Real polynomial multiplication is essentially self-adjoint on Schwartz: compactly supported smooth functions already form a core for its maximal multiplication realization. Lemma 2 shows that $e^{itc q_1^d} \in G$ for all real t , since each e^{itA_s} is an allowed conjugated evolution with a rescaled real time. Thus all q_1^d phase gates belong to G . We have not assumed that the original evolution e^{itH^*} preserves Schwartz space.

4 Cubic phases, quartic phases, and a separating drift

Gaussian translations conjugate q_1^d to $(q_1 + a)^d$. Their multiplication phase gates commute. Applying $d - 3$ unit-step finite differences therefore produces the phase of a cubic polynomial with nonzero leading coefficient. Its lower-degree part is Gaussian and commutes with it, so all $e^{itq_1^3}$ belong to G .

Write $q = q_1, p = p_1$ and define self-adjoint conjugates $B_{\pm} = e^{\pm i q^3} p^2 e^{\mp i q^3}$. Cubic multiplication phases preserve Schwartz, and on that space

$$e^{\pm i q^3} p e^{\mp i q^3} = p \mp 3q^2, \quad B_+ + B_- - 2p^2 = 18q^4. \quad (4.1)$$

The sum on Schwartz is essentially self-adjoint polynomial multiplication. Lemma 3 implies $e^{itq^4} \in G$. Metaplectic covariance then gives $e^{itL^4} \in G$ for every real linear form L in the canonical coordinates. The fourth power of a linear operator has Weyl symbol equal to the fourth power of the corresponding linear form.

Fourth powers of real linear forms span all homogeneous quartic polynomials, by polarization. Choose positive numbers $\alpha_1, \dots, \alpha_n$ linearly independent over $\mathbb{Q}$, and let

$$F = \left(\sum_{j=1}^n \alpha_j N_j \right)^2. \quad (4.2)$$

Its Weyl symbol has degrees four, two, and zero; thus it is a finite real linear combination of L^4 terms and Gaussian Hamiltonians. Let $D = \text{span}\{|k\rangle : k \in \mathbb{N}_0^n\}$ be the finite Fock span. On D , F is diagonal with eigenvalues $f(k) = (\sum_j \alpha_j k_j)^2$. It is essentially self-adjoint there, since finite sequences form a core for the maximal real diagonal operator. Each summand in the polynomial expression for F is self-adjoint in its natural realization and has D in its domain. Lemma 3, on this core, proves $e^{itF} \in G$; below F denotes its self-adjoint closure.

5 Spectral filters with controlled domains

Fix a mode j and define the self-adjoint conjugate $Q_j(t) = e^{itF} q_j e^{-itF}$. It leaves D invariant. Its raising matrix element from k to $k + e_j$ is

$$a_k e^{it\omega_j(k)}, \quad a_k = \sqrt{(k_j + 1)/2}, \quad \omega_j(k) = 2\alpha_j \sum_l \alpha_l k_l + \alpha_j^2. \quad (5.1)$$

For fixed j these frequencies are positive and pairwise distinct: equality implies $\sum_l \alpha_l (k_l - k'_l) = 0$ and hence $k = k'$. Separation between frequencies of different modes is unnecessary.

Fix k and $\omega = \omega_j(k)$. For positive integers L , define on D

$$C_L = \frac{2}{L^3} \sum_{r=0}^{L^3-1} \cos(\omega r/L^2) Q_j(r/L^2). \quad (5.2)$$

We first justify the use of its exponential. For any finite real sum $B = \sum_l b_l Q_j(t_l)$ put $W = \sum_l |b_l|$. If ψ has total-number support at most M , then $B\psi$ has support at most $M+1$ and

$$\|B\psi\| \leq W \sqrt{2(M+1)} \|\psi\|, \quad \|B^s \psi\| \leq (W\sqrt{2})^s \sqrt{\frac{(M+s)!}{M!}} \|\psi\|. \quad (5.3)$$

The first estimate uses the usual creation and annihilation bounds; e^{itF} preserves each photon cutoff. The second follows by induction. For every finite z , the series $\sum_s \|B^s \psi\| z^s / s!$ converges by the ratio test. Every vector in the dense invariant space D is therefore analytic for B . Nelson's analytic-vector theorem implies essential self-adjointness on D [4, Theorem 1.7]. In particular this applies to C_L , for which $W \leq 2$. The evolution of each $Q_j(t_l)$ belongs to G , as a conjugate of a Gaussian evolution. Lemma 3 now yields $e^{itC_L} \in G$.

For a fixed basis vector only finitely many adjacent matrix elements occur. Each filter coefficient is a Riemann-sum approximation to

$$\frac{2}{L} \int_0^L \cos(\omega t) e^{\pm i\nu t} dt. \quad (5.4)$$

With mesh L^{-2} , its error is $O((\omega + |\nu|)L^{-2})$ for fixed frequencies. For positive ν , the integral tends to 1 if $\nu = \omega$ and 0 otherwise. Consequently $C_L \psi \rightarrow X_{j,k} \psi$ on D , where

$$X_{j,k} = a_k(|k + e_j\rangle\langle k| + |k\rangle\langle k + e_j|). \quad (5.5)$$

This limit is bounded self-adjoint; the dense space D is a core for it. Lemma 2 therefore gives $e^{itX_{j,k}} \in G$. Replacing cosine by sine gives the same estimates and the bounded limit

$$Y_{j,k} = ia_k(|k + e_j\rangle\langle k| - |k\rangle\langle k + e_j|), \quad e^{itY_{j,k}} \in G. \quad (5.6)$$

Indeed the averages of $2 \sin(\omega t) e^{\pm i\omega t}$ tend to $\pm i$. Thus bounded single-edge Hamiltonians have been obtained without an unbounded commutator formula or an assumption that compressed evolutions equal compressed generators.

6 Strong density in the unitary group

On a finite connected box of Fock vertices, the edge operators $X_{j,k}$ and $Y_{j,k}$ generate all special unitaries on the box, extended by the identity outside. To see this, the commutator of each edge pair yields the difference of its two endpoint projections. Commutators along paths produce off-diagonal matrix units between any vertices. Together these span the traceless skew-Hermitian matrices. Every operator used at this stage is bounded and supported on the finite box, so finite-dimensional Lie product formulas apply inside the closed group G .

These finite-support special unitaries are strongly dense in the full unitary group. Fix finitely many Fock basis input vectors and a target unitary U . Project their U -images into growing connected boxes that contain the input vectors. The Gram matrices tend to the identity, so multiplication by their inverse square roots orthonormalizes the projected images without changing their limits. Extend the resulting isometry to a unitary of the box. Choose the box large enough to leave at least one unconstrained input column; multiply that column by a phase to correct the determinant. Extend by the identity outside the box. This special unitary approximates U on the prescribed input vectors. Approximation of arbitrary vectors by finite Fock expansions proves strong density. Hence G is the full unitary group, and every strong neighborhood of U contains an actual finite allowed product.

7 Uniform approximation with an arbitrary reference

Let P_K project onto total photon number at most the nonnegative integer K . It has finite rank. For any admissible ρ_{AR} , set $P = P_K \otimes I_R$ and $\sigma = P\rho_{AR}P$. The discarded weight obeys

$$\eta = \text{Tr}[(I - P)\rho_{AR}] \leq \frac{E}{K + 1}. \quad (7.1)$$

The projection gentle-measurement estimate gives $\|\rho_{AR} - \sigma\|_1 \leq 2\sqrt{\eta}$. For completeness, it follows first for a purification by direct calculation of the trace norm of a rank-two difference, which is $\sqrt{4\eta - 3\eta^2} \leq 2\sqrt{\eta}$, and then for the original state by partial trace. It is uniform in R .

For an allowed product V , put $\delta = \|(U - V)P_K\|$. Expanding the difference of conjugations on the subnormalized state σ gives a trace-norm bound 2δ . On the Hermitian remainder, each unitary channel preserves trace norm. Thus

$$d_E(U, V) \leq 2\|(U - V)P_K\| + 4\sqrt{\frac{E}{K + 1}}. \quad (7.2)$$

If $E > 0$, choose K so that the second term is less than $\varepsilon/2$; if $E = 0$, it vanishes already. Strong density and finite rank of P_K give an actual finite allowed product with $\delta < \varepsilon/4$. Equation (7.2) is then strictly less than ε , proving Theorem 1.

All statements are qualitative. The construction permits arbitrarily large squeezing, positive and negative evolution times, and intermediate energies without a stated bound. It does not imply an efficient compiler, a polynomial gate count, or a finite-precision resource estimate. Finite algebraic and filter diagnostics in the source archive do not certify the infinite-dimensional theorem; that conclusion rests on the core, product formula, and limiting arguments above.

Verification, provenance, and scope

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