Five settings suffice opposite three binary measurements

Yuxuan Zhang

Version 1.1 — 19 November 2026

Abstract

We prove a finite-setting criterion for bipartite Bell locality. If one party has measurements with outcome counts $a_1, \dots, a_s$, a no-signalling behavior is local if and only if every restriction to at most $t = \prod_x a_x - \sum_x (a_x - 1)$ settings of the other party is local. The proof applies Helly's theorem to the compatible hidden-variable distributions with fixed observed marginals. For three binary measurements the bound is five, improving the sixteen-setting reduction of Budroni and Cabello. Combined with the known complete finite facet list, this proves the conjecture of Jesus and Zambrini Cruzeiro for all $(N, 3, 2, 2)$ scenarios, $N \geq 5$. The bound five is sharp. We also provide an exact rational nonlocal behavior whose every four-setting restriction is local, together with explicit local decompositions and a verifier.

Research draft. A separate AI reviewer has checked the proof and sources.
External specialist review and priority confirmation remain pending.

1 The finite-setting question

A complete list of Bell facets describes all local behaviors in a given measurement scenario. Increasing the number of settings enlarges both the probability space and the collection of deterministic behaviors, so a complete finite list does not by itself imply a classification for arbitrarily many settings.

Budroni and Cabello reduced this problem to compatibility of partial classical probability distributions [1]. For three binary measurements on one side, their Fourier–Motzkin argument gives a bound of sixteen settings on the other side. Jesus and Zambrini Cruzeiro established the complete facet list of $(6, 3, 2, 2)$ and found the same six nontrivial classes as at $(5, 3, 2, 2)$ [2, Section IV.B and Appendix B.2]. They conjectured that the list remains complete for every $(N, 3, 2, 2)$ with $N \geq 5$.

We obtain the arbitrary-N conclusion by applying Helly's theorem to the convex sets of compatible hidden distributions. The general compatibility criterion, including arbitrary finite outcome sets, is Pironio's Lemma 5 in Ref. [3]. Once the three one-measurement marginals are fixed, a distribution on the eight deterministic assignments has only four free affine coordinates. Helly's theorem then makes five settings sufficient. Both the compatible-extension description and the finite facet classes are inherited. The contribution is the sharper setting bound obtained from standard convex geometry and its consequence for the conjecture. For two binary measurements on one side, the stronger full CHSH classification is already Theorem 1 of Ref. [3].

2 A locality criterion from the marginal fiber

Alice chooses $x \in \{1, \dots, s\}$ and obtains an outcome in a finite set A_x, with $|A_x| = a_x \geq 2$. Bob chooses y in a nonempty finite set Y and obtains an outcome in a finite nonempty set B_y. A

behavior $p(a,b \mid x,y)$ is assumed nonnegative, normalized for each (x,y), and no-signalling. We write its one-party marginals as $p_A(a \mid x)$ and $p_B(b \mid y)$. It is local if it admits a decomposition

$$p(a,b \mid x,y) = \sum_{\lambda} w_\lambda r_A(a \mid x,\lambda) r_B(b \mid y,\lambda), \qquad w_\lambda \geq 0, \quad \sum_{\lambda} w_\lambda = 1. \tag{2.1}$$

The response distributions may be made deterministic by refining the hidden variable. For $S \subseteq Y$, let $p|_S$ denote the restriction to Bob's settings in S.

Theorem 1. *Set*

$$t = \prod_{x=1}^{s} a_x - \sum_{x=1}^{s} (a_x - 1). \tag{2.2}$$

Then p is local if and only if $p|_S$ is local for every $S \subseteq Y$ with $1 \leq |S| \leq t$.

Proof. Let $\Omega = \prod_x A_x$ be Alice's deterministic assignments. An element $\alpha \in \Omega$ prescribes the outcome α_x for every measurement. Choose a reference outcome a_x^0 in each A_x. Consider the affine space

$$E_p = \left\{ \lambda \in \mathbb{R}^\Omega : \sum_{\alpha} \lambda_\alpha = 1, \quad \sum_{\alpha : \alpha_x = a} \lambda_\alpha = p_A(a \mid x) \ \ (a \neq a_x^0) \right\}. \tag{2.3}$$

The equations in (2.3) are independent. Indeed, if a constant plus a sum of functions of the separate coordinates α_x vanishes on Ω, with each function zero at its reference outcome, evaluation at the all-reference assignment sets the constant to zero. Varying one coordinate then sets each remaining coefficient to zero. The product distribution $\prod_x p_A(\alpha_x \mid x)$ belongs to E_p, so

$$\dim E_p = |\Omega| - 1 - \sum_{x} (a_x - 1) = t - 1. \tag{2.4}$$

This is an affine dimension calculation; nonnegativity may restrict feasible distributions to a smaller face when observed marginals vanish.

For each $y \in Y$, let $C_y \subseteq E_p$ consist of the vectors λ that admit nonnegative numbers $q_y(\alpha, b)$ satisfying

$$\sum_{b \in B_y} q_y(\alpha, b) = \lambda_\alpha, \tag{2.5}$$

$$\sum_{\alpha : \alpha_x = a} q_y(\alpha, b) = p(a,b \mid x,y) \quad \text{for all } x, a, b. \tag{2.6}$$

Each C_y is a convex compact polytope, being the projection of a nonnegative normalized polytope of joint arrays. No-signalling ensures that the same Alice marginal constraints occur for every y.

The following compatibility equivalence is the finite-subfamily version of Ref. [3, Lemma 5]; we include its proof to make the zero-weight case explicit. For any nonempty $S \subseteq Y$,

$$p|_S \text{ is local} \quad \Longleftrightarrow \quad \bigcap_{y \in S} C_y \neq \varnothing. \tag{2.7}$$

A local model deterministic on Alice supplies a distribution λ on Ω and joint distributions q_y satisfying (2.5)–(2.6), proving the forward direction.

Conversely, choose a common $\lambda \in \bigcap_{y \in S} C_y$ and the corresponding arrays. For $\lambda_\alpha > 0$, define $r_y(b \mid \alpha) = q_y(\alpha, b)/\lambda_\alpha$. If $\lambda_\alpha = 0$, nonnegativity in (2.5) forces $q_y(\alpha, b) = 0$ for all b; choose any normalized r_y on that atom. Equation (2.6) becomes

$$p(a,b \mid x,y) = \sum_{\alpha} \lambda_\alpha \mathbf{1}\{a = \alpha_x\} r_y(b \mid \alpha). \tag{2.8}$$

This is one local model for all $y \in S$. A fully deterministic refinement has weights

$$w(\alpha, (b_y)_{y\in S}) = \lambda_\alpha \prod_{y\in S} r_y(b_y \mid \alpha). \tag{2.9}$$

These weights sum to one and recover (2.8).

Locality implies locality of every restriction. For the reverse implication, the hypothesis and (2.7) say that every subfamily of at most t sets C_y has nonempty intersection. Helly's theorem in the affine space of dimension $t-1$ in (2.4) gives $\bigcap_{y\in Y} C_y \neq \varnothing$. Equation (2.7) proves the result. If $|Y| < t$, the assumption already includes the full set of settings. □

For reference, the finite form of Helly used here follows from Radon's lemma. Given $D+2$ convex sets in $\mathbb{R}^D$ whose every $D+1$ intersect, choose z_i in every set except the ith. A Radon partition of the $D+2$ points has intersecting convex hulls. For each convex set, one of the two hulls uses only points in that set, so their common point belongs to all the sets. Induction on the size of the family, replacing two sets by their intersection, proves the general finite statement. No interior-point assumption is required.

3 Facet completeness for $(N, 3, 2, 2)$

Write $\mathcal{L}_{A,Y}$ for the local polytope in the full conditional probability space, and π_S for restriction to Bob's settings in S. All affine hulls below include normalization and no-signalling.

Corollary 2. *With t as in (2.2),*

$$\mathcal{L}_{A,Y} = \bigcap_{\substack{S\subseteq Y \\ 1\le|S|\le t}} \pi_S^{-1}(\mathcal{L}_{A,S}). \tag{3.1}$$

Every facet of $\mathcal{L}_{A,Y}$ is supported on at most t Bob settings, up to affine identities.

Proof. Any point on the right side is nonnegative on every setting, since each setting lies in a tested restriction. Theorem 1 therefore gives (3.1). Pull back a finite facet description of each $\mathcal{L}_{A,S}$. This yields a finite half-space description of the right side. At a relative-interior point of any facet, at least one defining inequality is active. Its hyperplane contains the entire facet and therefore is the facet's supporting hyperplane. Every such defining inequality involves at most t settings. □

The local polytopes are full-dimensional in their no-signalling affine hulls. One can see this by choosing all but one outcome indicator for each measurement. The constant, single-party indicators and products of one indicator from each party are linearly independent functions of the deterministic assignments. These are precisely the independent no-signalling coordinates.

Theorem 3. *For every integer $N \geq 5$, all facets of the full-probability local polytope in scenario $(N, 3, 2, 2)$ are setting liftings and relabelings of positivity and the following six nontrivial classes:*

$$\text{CHSH}, \qquad I_{3322}, \qquad I^1_{4322},\ I^2_{4322},\ I^3_{4322}, \qquad I_{5322}. \tag{3.2}$$

Proof. For three binary measurements, (2.2) gives $t = 2^3 - 3 = 5$. Corollary 2 reduces membership and all facet inequalities to the $(5, 3, 2, 2)$ restrictions, after exchanging the parties. The complete finite list is positivity and (3.2), as given in Ref. [2, Section IV.B and Appendix B.2]. Smaller restrictions add no further classes, since their inequalities lift to the five-setting polytope. Setting lifting preserves facets [5]; a direct proof of the binary case appears in Appendix A. Thus every class in (3.2) remains a facet for all $N \geq 5$, and no additional class is needed. □

The finite enumeration in this argument is a result of Ref. [2]; it is not repeated here. The new reduction removes the unbounded setting count from its completeness question. In particular, it applies to the full probability polytope, including the one-party marginals.

4 Why four settings do not suffice

The known genuine five-setting facet I_{5322} already shows that Theorem 1 is sharp when Alice has three binary measurements. Choose a point in the relative interior of that facet and perturb it slightly outward. Every other facet inequality remains strict. An inequality supported on four settings cannot define this genuine five-setting facet. Taking the perturbation sufficiently small therefore gives a nonlocal behavior whose every four-setting restriction is local.

Here is an explicit rational control. Use binary coordinates

$$A_x = p_A(1 \mid x), \qquad B_y = p_B(1 \mid y), \qquad C_{xy} = p(1,1 \mid x,y), \tag{4.1}$$

where $x = 1,2,3$ and $y = 1,\dots,5$. Let

$$H = \begin{pmatrix} -1 & -1 & 0 & 1 & 1 \\ 1 & 0 & -1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}. \tag{4.2}$$

The inequality

$$F(A,B,C) := 3 - A_2 - 2A_3 - B_4 - 2B_5 + \sum_{x,y} H_{xy} C_{xy} \geq 0 \tag{4.3}$$

is a relabeling of I_{5322} in Ref. [4], Table A.4, and is valid on every local behavior. For a deterministic Alice assignment $a = (a_1, a_2, a_3)$, minimization over Bob's bits gives

$$\min_b F(a,b,ab) = 3 - a_2 - 2a_3 + \sum_{y=1}^{5} \min\{0, v_y(a)\} = 0, \tag{4.4}$$

$$v(a) = (-a_1 + a_2,\ -a_1 + a_3,\ -a_2 + a_3, \\ -1 + a_1 + a_2 + a_3,\ -2 + a_1 + a_2 + a_3). \tag{4.5}$$

The equality is checked on the eight binary triples a.

Define a behavior by

$$A = \frac{1}{10000}(5000, 5001, 5002), \tag{4.6}$$

$$B = \frac{1}{10000}(5000, 5000, 5000, 3501, 6502), \tag{4.7}$$

$$C = \frac{1}{10000} \begin{pmatrix} 3501 & 3501 & 2500 & 499 & 1999 \\ 1499 & 2500 & 3501 & 499 & 1999 \\ 2500 & 1499 & 1499 & 499 & 1999 \end{pmatrix}. \tag{4.8}$$

For each (x,y), the four probabilities are C_{xy}, $A_x - C_{xy}$, $B_y - C_{xy}$ and $1 - A_x - B_y + C_{xy}$. They sum to one, are no-signalling by construction, and are all at least $99/2000$. Nevertheless,

$$F(A,B,C) = -\frac{11}{5000}. \tag{4.9}$$

The accompanying certificate gives, for each deleted Bob setting, twenty nonnegative rational weights on deterministic local behaviors that recover the remaining four-setting behavior exactly. Thus every four-setting restriction is local.

The verifier uses only Python's standard library and exact rational arithmetic. It checks all 256 deterministic vertices, the 40 vertices saturating (4.3), and their affine rank 22 in the 23-dimensional polytope. It also checks normalization, positivity, all five local decompositions, and a six-setting gluing control with null hidden-variable weights. Three deliberately altered certificates are rejected. These finite checks support the explicit example; the arbitrary-N conclusion follows from the analytic proof.

5 Scope

The criterion concerns locality of finite bipartite behaviors and allows arbitrary finite outcome sets for Bob. For binary Alice measurements the general upper bound is $2^s - s$. Its optimality for $s \geq 4$ is not established here. Nor does the explicit no-signalling example assert a quantum realization. The result classifies the asymmetric family with three binary measurements on one side, rather than general Bell polytopes with unrestricted measurement counts on both sides.

A Facet preservation under a new binary setting

For s Alice and $n \geq 1$ Bob binary settings, the independent coordinates in (4.1) have dimension $D = s + n + sn$. A deterministic vertex has the form $(a, b, (a_x b_y)_{xy})$. The constant and these coordinate monomials are linearly independent on the binary cube.

Let G be a facet. The Alice assignments among its saturating vertices affinely span $\mathbb{R}^s$. Otherwise a nonzero affine function of the Alice coordinates vanishes on G. Since G has codimension one, its supporting hyperplane would depend on Alice alone. A nonconstant linear function on the Alice cube attains its extremum only where at least one bit a_x is fixed. On the corresponding local face, the n coordinates C_{xy} would then all be zero or equal to B_y, in addition to the fixed A_x. These $n + 1$ independent constraints contradict the facet dimension.

Append one Bob setting and lift the inequality by zero coefficients on its new coordinates. Each old saturating vertex has two saturating extensions, with the new bit equal to zero and one. Their difference is supported on the $s + 1$ new coordinates and equals $(1, a_1, \ldots, a_s)$. These differences span all new directions. The extensions with new bit zero retain the old affine dimension $D - 1$, so the lifted face has dimension $D - 1 + s + 1$, one below the new polytope dimension. It is a facet.

Reproducibility and review status

The supplementary package contains the rational certificate, `verify_control.py`, and a hash-checked entry point `reproduce.py`. Running `python3 reproduce.py` from an extracted package checks the explicit example without network access or an optimization solver. A separate AI reviewer, given the frozen manuscript in fresh context, reconstructed the decisive steps, implemented independent exact checks, and compared the primary sources. This is an additional internal review, not external specialist validation. The reviewer also checked the six source-facet orbits against Ref. [4]; the finite-list completeness itself remains an imported result. The package includes the review and its limits. No prior statement of the five-setting conclusion was located in that targeted comparison, but first priority is not confirmed.

References

[1] C. Budroni and A. Cabello, Bell inequalities from variable elimination methods, *J. Phys. A: Math. Theor.* **45**, 385304 (2012). doi:10.1088/1751-8113/45/38/385304; arXiv:1112.5876v2.

[2] J. Jesus and E. Zambrini Cruzeiro, Tight Bell inequalities from polytope slices, *Phys. Rev. A* **108**, 052220 (2023). doi:10.1103/PhysRevA.108.052220; arXiv:2212.03212v3.

[3] S. Pironio, All CHSH polytopes, *J. Phys. A: Math. Theor.* **47**, 424020 (2014). doi:10.1088/1751-8113/47/42/424020; arXiv:1402.6914v2.

[4] J. Diogo da Costa Jesus, *Semi-device Independent Protocols for Quantum Key Distribution*, M.Sc. thesis in Engineering Physics, Instituto Superior Técnico (December 2022), Section 5.4 and Table A.4. Institutional repository.

[5] S. Pironio, Lifting Bell inequalities, *J. Math. Phys.* **46**, 062112 (2005). doi:10.1063/1.1928727; arXiv:quant-ph/0503179v3.