One ebit is necessary and sufficient for the elegant joint measurement

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Abstract

We determine the exact deterministic LOCC entanglement cost of the two-qubit elegant joint measurement when its output is classical and the shared pure resource can have any finite Schmidt rank. The cost is one ebit. The proof combines a coherent-detector bound $\lambda_1 \leq 2/3$ with a dimension-independent constraint on the two largest resource Schmidt probabilities and their tail: $(\lambda_1 - \lambda_2)^2 \leq 4\delta(1-\delta)$, where $\delta = \sum_{a\geq 3} \lambda_a$. The latter follows from a two-dimensional adjugate identity and completeness of the entire separable measurement. Together these constraints force resource entropy at least one, with equality only for a Bell-pair spectrum. Teleportation attains the bound.

1 The question and the result

A joint measurement on separated quantum systems can require more entanglement to implement than is present in its measurement eigenstates. The elegant joint measurement (EJM) is a useful instance: its four two-qubit eigenstates have equal, nonmaximal entanglement and tetrahedral local marginals [1]. One shared Bell pair implements the measurement by teleportation. The question is whether a pure resource with less than one ebit could suffice if its local dimension is unrestricted.

Akibue, Miyazaki and Osaka proved that a pure resource of the same Schmidt rank as an entangled state must be maximally entangled to implement its exact verification measurement by separable operations [2, Theorem 3 and Corollary 1]. Their result settles Schmidt-rank-two resources for the EJM, but Section 5.3 leaves higher resource rank open. Bandyopadhyay gives an elementary proof of the matched-rank result and retains the EJM question in Section 5.2 [3]. The higher-rank issue matters: other entangled two-qubit bases admit exact implementations using higher-rank resources with less than one ebit [3, Section 5.1]. A rank-two necessity theorem alone therefore does not give an unrestricted entropy lower bound.

We use the tetrahedral vectors

$$n_1 = \frac{(1,1,1)}{\sqrt{3}}, \quad n_2 = \frac{(1,-1,-1)}{\sqrt{3}}, \quad n_3 = \frac{(-1,1,-1)}{\sqrt{3}}, \quad n_4 = \frac{(-1,-1,1)}{\sqrt{3}}. \tag{1}$$

With $|s\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ and $\boldsymbol{\sigma} = (X, Y, Z)$, define

$$|e_j\rangle = \left[\left(\tfrac{1}{2} I + \tfrac{\sqrt{3}}{2} n_j \cdot \boldsymbol{\sigma}\right) \otimes I\right] |s\rangle\,, \qquad P_j = |e_j\rangle\langle e_j|\,. \tag{2}$$

These states are orthonormal. The channel to be implemented is

$$\mathcal{M}_{\mathrm{EJM}}(\rho) = \sum_{j=1}^{4} \mathrm{Tr}(P_j \rho)\, |j\rangle\langle j|_{C_A} \otimes |j\rangle\langle j|_{C_B}\,. \tag{3}$$

Alice holds the input qubit A, Bob holds B, and C_A, C_B are classical registers. No quantum postmeasurement state is prescribed.

A pure resource $|\chi\rangle \in \mathbb{C}^r \otimes \mathbb{C}^r$ is admissible if a deterministic LOCC channel across $AA' : BB'$ satisfies $\Lambda(\rho \otimes |\chi\rangle\langle\chi|) = \mathcal{M}_{\mathrm{EJM}}(\rho)$ for every input density operator ρ. All supplied shared entanglement is included in χ. The cost is the infimum of $S(\chi_{A'})$ over admissible resources and all finite r. Entropies and logarithms in entropy expressions are base two.

Theorem 1. *The exact deterministic LOCC cost of the channel* (3) *is*

$$E_{\mathrm{LOCC}}^{\mathrm{exact}}(\mathcal{M}_{\mathrm{EJM}}) = 1. \tag{4}$$

If an admissible pure resource has entropy one, its nonzero Schmidt probabilities are $(1/2, 1/2)$.

The proof has two inputs. A coherent detector, using the measurement-cost method of Ref. [4], bounds the largest Schmidt probability. A separate argument uses all four measurement effects to control the resource's Schmidt tail. The latter is the step that removes the resource-rank restriction. Section 4 combines the two bounds.

The operational model here allows communication during the protocol. The entanglement costs studied in measurement localization without communication during the local measurement stage concern a different task [6].

2 A bound on the largest Schmidt probability

For a bipartite vector, its coefficient matrix has the Alice index as row and the Bob index as column; $\mathrm{vec}(E) = \sum_{i,k} E_{ik}\,|i\rangle\,|k\rangle$. Put

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad E_j = \frac{(I + \sqrt{3}\, n_j \cdot \boldsymbol{\sigma})J}{2\sqrt{2}}, \qquad U_j = (n_j \cdot \boldsymbol{\sigma})J. \tag{5}$$

Then $\mathrm{vec}(E_j) = |e_j\rangle$, $\mathrm{Tr}(E_j^\dagger E_k) = \delta_{jk}$, U_j is unitary, and $|\det E_j| = 1/4$.

Lemma 2. *Every admissible resource has largest Schmidt probability at most* $2/3$.

Proof. Give Alice an untouched reference qubit C and Bob an untouched reference qubit D. Consider the normalized test input

$$|\Psi\rangle_{ABCD} = \sum_{j=1}^{4} |e_j\rangle_{AB} \otimes \mathrm{vec}(D_j)_{CD}, \qquad D_j = \frac{\overline{U_j}}{2\sqrt{2}}. \tag{6}$$

Each D_j has squared Frobenius norm $1/4$, so exact EJM measurement on AB leaves a maximally entangled pair on CD in every outcome, with probability $1/4$. A local correction depending on j gives a fixed Bell pair $|\Phi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$.

Across the cut $AC : BD$, the coefficient matrix of Ψ is

$$M = \sum_j E_j \otimes D_j = \frac{1}{\sqrt{12}} \left(\sum_{a=x,y,z} \sigma_a \otimes \overline{\sigma_a} \right) (J \otimes \overline{J}). \tag{7}$$

Here we used $\sum_j n_j = 0$ and $\sum_j n_j n_j^{\mathsf{T}} = (4/3)I$. The operator in parentheses is $4\,|\Phi\rangle\langle\Phi| - I_4$, and the right factor is unitary. Thus the Schmidt probabilities of Ψ are

$$(3/4, 1/12, 1/12, 1/12). \tag{8}$$

Equality of the channels on every density operator is equality of linear maps, and hence also holds with untouched reference systems. An admissible implementation therefore converts $|\chi\rangle \otimes |\Psi\rangle$ deterministically to a Bell pair by LOCC, after correcting the outcome and discarding all other systems. The largest-component condition in pure-state majorization [5] gives $(3/4)\lambda_1 \le 1/2$, which is the claim. The entanglement of the test input is accounted for on the input side of this conversion. □

3 A resource-rank-independent tail constraint

Write the resource in Schmidt form,

$$|\chi\rangle = \sum_{a=1}^{r} \sqrt{\lambda_a}\,|aa\rangle\,, \qquad \lambda_1 \geq \lambda_2 \geq \cdots\,, \qquad L = \mathrm{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_r}). \tag{9}$$

A rank-one resource cannot induce an entangled rank-one measurement effect by separable operations. We may therefore assume $r \geq 2$, padding with zero coordinates if needed, and put

$$p = \lambda_1, \qquad q = \lambda_2, \qquad \delta = \sum_{a\geq 3} \lambda_a, \qquad s = p + q = 1 - \delta. \tag{10}$$

Proposition 3. *Every pure resource enabling an exact separable implementation of the EJM satisfies*

$$(p - q)^2 \leq 4(1 - \delta)\delta. \tag{11}$$

The following identity controls the error caused by the resource tail. For a two-by-two matrix K, let $K_0 = K - \frac{1}{2}\,\mathrm{Tr}(K)I$.

Lemma 4. *For arbitrary complex two-by-two matrices X, Y and $c \in \mathbb{C}$,*

$$c(YX)_0 = \big(Y\,\mathrm{adj}(cI - XY)X\big)_0. \tag{12}$$

Proof. Using $\mathrm{adj}(V) = \mathrm{Tr}(V)I - V$ in dimension two and writing $K = YX$, the expression before taking its traceless part is

$$Y\,\mathrm{adj}(cI - XY)X = (c - \mathrm{Tr}\,K)K + K^2 = cK - \det(K)I.$$

The last equality is Cayley–Hamilton. □

Proof of Proposition 3. An exact separable implementation has effects

$$S_j = \sum_t |a_{jt}\rangle\langle a_{jt}| \otimes |b_{jt}\rangle\langle b_{jt}|\,, \qquad \sum_{j=1}^{4} S_j = I_{AA'BB'}. \tag{13}$$

The product vectors are unnormalized. Finite decompositions exist for finite-dimensional separable effects. This necessary representation also applies to LOCC with arbitrary communication rounds and to limits of such protocols. Local ancillas initially in product states can be absorbed into the effects.

Let A_{jt}, B_{jt} be the 2-by-r coefficient matrices of a_{jt}, b_{jt}. Contraction with the resource induces the rank-one effect P_j. Positivity then gives

$$A_{jt} L B_{jt}^{\mathsf{T}} = c_{jt} E_j, \qquad \sum_t |c_{jt}|^2 = 1. \tag{14}$$

Let v_j be $e_j \otimes \chi$, reordered as $AA' : BB'$. Since $\langle v_j|\,S_j\,|v_j\rangle = 1$ and $0 \leq S_j \leq I$, we have

$$v_j = S_j v_j = \sum_t \overline{c_{jt}}\,a_{jt} \otimes b_{jt}. \tag{15}$$

These contraction and eigenvector reductions also underlie the matched-rank proof in Ref. [3].

Temporarily suppress j, t. Split $A = [A_0\ A_{\mathrm{t}}]$, $B = [B_0\ B_{\mathrm{t}}]$ and $L = L_0 \oplus L_{\mathrm{t}}$, where $L_0 = \mathrm{diag}(\sqrt{p}, \sqrt{q})$, and define

$$T = A_{\mathrm{t}} L_{\mathrm{t}} B_{\mathrm{t}}^{\mathsf{T}}, \qquad Q = A_0 L_0 Z B_0^{\mathsf{T}}, \qquad f_E(A, B) = \mathrm{Tr}\Big(Z B_0^{\mathsf{T}} E^{-1} A_0 L_0\Big)\,. \tag{16}$$

Here $Z = \mathrm{diag}(1, -1)$. Apply Lemma 4 with $X = E^{-1}A_0L_0$, $Y = B_0^{\mathsf{T}}$, and $cI - XY = E^{-1}T$. The adjugate product identity $\mathrm{adj}(E^{-1}T) = \mathrm{adj}(T)E/\det E$ gives

$$cf_E(A, B) = \frac{\mathrm{Tr}(\mathrm{adj}(T)Q)}{\det E}. \tag{17}$$

This identity does not divide by c or assume that A_0, B_0 are invertible. Since $\|\mathrm{adj}(T)\|_{\mathrm{F}} = \|T\|_{\mathrm{F}}$,

$$|cf_E(A, B)| \le \frac{\|T\|_{\mathrm{F}}\,\|Q\|_{\mathrm{F}}}{|\det E|}. \tag{18}$$

The bilinear expression $f_E(A, B)$ defines a complex-linear functional on the tensor $a \otimes b$. Explicitly, its value on a tensor w in the local ordering is

$$f_E(w) = \sum_{a=1}^{2} Z_{aa}\sqrt{\lambda_a} \sum_{i,k=0}^{1} (E^{-1})_{ki}\, w_{ia;ka}. \tag{19}$$

The components of v_j are $(v_j)_{ia;kb} = (E_j)_{ik}\sqrt{\lambda_a}\,\delta_{ab}$. Consequently

$$f_{E_j}(v_j) = 2(p - q). \tag{20}$$

Apply this functional to (15), take absolute values, and use (18). Because $|\bar{c}\,f| = |cf|$, for each outcome

$$2(p - q)|\det E_j| \le \sum_t \|T_{jt}\|_{\mathrm{F}}\,\|Q_{jt}\|_{\mathrm{F}}\,. \tag{21}$$

Completeness in (13) fixes both global squared-norm sums:

$$\sum_{j,t} \|T_{jt}\|_{\mathrm{F}}^2 = 4\delta, \qquad \sum_{j,t} \|Q_{jt}\|_{\mathrm{F}}^2 = 4s. \tag{22}$$

For the first equality, contract the identity with $|ik\rangle_{AB} \otimes \sum_{a\ge3} \sqrt{\lambda_a}\,|aa\rangle_{A'B'}$ and sum over the four choices of i, k. For the second, use the signed vector $\sqrt{p}\,|11\rangle - \sqrt{q}\,|22\rangle$ in place of the tail. The product terms with $c_{jt} = 0$ remain included in these sums.

Finally, $\sum_j |\det E_j| = 1$. Sum (21) over all four outcomes and apply Cauchy–Schwarz and (22):

$$2(p - q) \le \left(\sum_{j,t} \|T_{jt}\|_{\mathrm{F}}^2\right)^{1/2} \left(\sum_{j,t} \|Q_{jt}\|_{\mathrm{F}}^2\right)^{1/2} = 4\sqrt{s\delta}.$$

Squaring proves (11). □

At $\delta = 0$, Proposition 3 recovers $p = q$. For larger resource rank it quantitatively limits the imbalance of the two leading probabilities. The estimate sums over the entire measurement; its constant does not grow with the resource dimension.

4 Entropy and optimality

Proof of Theorem 1. Let λ be the Schmidt probability vector of an admissible resource. By Lemma 2, $p \le 2/3$. If $p \le 1/2$, the vector $(1/2, 1/2, 0, \ldots)$ majorizes λ, so $H(\lambda) \ge 1$.

Suppose instead that

$$p = \tfrac{1}{2} + x, \qquad 0 < x \le \tfrac{1}{6}, \qquad v = 1 - p.$$

If $q \le v/2$, every entry of the normalized remaining probability vector is at most $1/2$. Its entropy is at least one, and therefore

$$H(\lambda) \ge h_2(p) + v \ge 2v + v \ge 1.$$

The binary-entropy chord bound $h_2(p) \ge 2(1-p)$ is strict for $1/2 < p < 1$, so the resulting entropy is strictly greater than one.

If $q > v/2$, then $\delta = v - q < v/2$ and $p - q = 2x + \delta$. Proposition 3 implies $4x^2 \le (2x+\delta)^2 \le 4(1-\delta)\delta \le 4\delta$, hence $\delta \ge x^2$. Merging all tail probabilities only decreases entropy. Monotonicity of h_2 on $[0, 1/2]$ now gives

$$\begin{aligned} H(\lambda) &\ge h_2(p) + v h_2(\delta/v) \\ &\ge h_2(p) + v h_2(x^2/v) \\ &\ge h_2(p) + x^2 \log_2(v/x^2). \end{aligned} \tag{23}$$

Here $v \ge 1/3$ and $x^2 \le 1/36$, so $v/x^2 \ge 12$. On $[1/2, 2/3]$,

$$-h_2''(u) = \frac{1}{\ln(2) u(1-u)} \le \frac{9}{2\ln 2}.$$

Taylor's integral remainder at $1/2$ thus gives $h_2(1/2 + x) \ge 1 - 9x^2/(4\ln 2)$. Substitution in (23) yields

$$H(\lambda) \ge 1 + \frac{\ln 12 - 9/4}{\ln 2} x^2 > 1. \tag{24}$$

The coefficient is positive without a numerical estimate: $\ln 3 = 2\,\mathrm{atanh}(1/2) > 1$ and $\ln 2 = 2\,\mathrm{atanh}(1/3) > 2/3$ imply $\ln 12 > 7/3 > 9/4$.

We have proved $H(\lambda) \ge 1$ for every finite resource rank. In the case $p \le 1/2$, equality requires exactly two probabilities equal to $1/2$; for $p > 1/2$ it is impossible by the preceding argument. Finally, a Bell pair lets Alice teleport her input qubit to Bob, who performs the EJM and communicates its outcome. This implements (3) at cost one and completes the proof. □

The bound holds for each finite rank before taking the infimum. A sequence of resources with increasing Schmidt rank therefore cannot lower the cost below one. The theorem concerns the supplied pure-resource entropy for an exact single implementation; it makes no assertion about approximate, asymptotic net-consumption, or communication-free localization costs.

Reproducibility and review status

The argument above is analytic. The accompanying source package checks the detector and tail identities in exact arithmetic. It includes 37 detector checks, 90 algebraic and physical-protocol checks, and a second implementation of the cleared-denominator identity (17) using coefficientwise rational polynomial arithmetic without SymPy. A physical rank-three positive control uses probabilities $(1/2, 1/3, 1/6)$, Bell-pair concentration and teleportation; its full separable POVM includes zero-contraction terms. Negative controls detect a wrong transpose, an omitted completeness contribution, and a wrong sign in the polynomial identity.

The derivation, source comparison, computation and drafting used AI assistance under the author's direction. The proof and package underwent sequential internal review. They have not received independent expert review or formal proof-assistant verification. The literature comparison identifies the higher-rank question left open in Refs. [2, 3]; it is not an exhaustive priority survey.

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