Eight-copy universal CCZ concentration beyond the two-thirds candidate optimum

Yuxuan Zhang

Institute of Physics, École Polytechnique Fédérale de Lausanne (EPFL),
CH-1015 Lausanne, Switzerland

Department of Physics, Princeton University,
Princeton, New Jersey 08544, USA

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Abstract

Rizzo and Leone gave a fixed stabilizer protocol that converts eight copies of an arbitrary unknown pure qubit ψ into an exact $|\mathrm{CCZ}\rangle$ state with probability $\frac{2}{3}m_3(\psi)$, where $m_3 = (1-x^6-y^6-z^6)/2$ in terms of the Bloch vector (x, y, z), and proved that no such protocol exceeds $\frac{7}{6}m_3$; whether $\frac{2}{3}m_3$ is optimal was left open. It is not. We give an explicit catalyst-free protocol of Pauli measurements, classical feedforward, Clifford unitaries and discarding whose nine accepted branches each output exactly $|\mathrm{CCZ}\rangle$, with total acceptance probability exactly $m_3(\psi)$ for every pure ψ. Hence the optimal eight-copy success probability lies in $[m_3, \frac{7}{6}m_3]$, and the candidate optimum $\frac{2}{3}m_3$ fails at every non-stabilizer input; at the T-type magic state the success probability rises from 8/27 to 4/9. Every accepted branch acts with rank one on the symmetric subspace, and a new branch recycles weight that a two-thirds protocol of the same form discards, raising the kept share of each of three sector directions from 4/7 to 6/7. A standalone program verifies every finite identity exactly, using only the Python standard library.

1 Introduction

Magic states supply the non-Clifford resource that makes Clifford operations universal [3]. In *universal* magic state concentration [1], a fixed stabilizer protocol, independent of the input, receives k copies of an arbitrary unknown pure qubit ψ and must output an exact magic state on every accepted branch; stabilizer protocols consist of Clifford unitaries, pure stabilizer auxiliary states, Pauli measurements, classical feed-forward and discarding. The relevant magic measure is $m_3(\psi) = \frac{1}{2}(1 - x^6 - y^6 - z^6)$, where (x, y, z) is the Bloch vector of ψ; in [1] it is denoted M_3^{lin}, a linearised order-three stabilizer Rényi entropy [4]. For eight copies and the target $|\mathrm{CCZ}\rangle$, Rizzo and Leone constructed a protocol with success probability $\frac{2}{3}m_3$ [1, Theorem 2, proved as Theorem 7] and proved that no stabilizer protocol exceeds $\frac{7}{6}m_3$ [1, Proposition 1, proved as Proposition 3, Eq. (133)]. Their six-copy protocol is optimal; at eight copies, optimality was left open.

The question is recorded in [2]. Let $\mathcal{U}_8$ be the class of fixed protocols built from stabilizer ancillas, Clifford unitaries, Pauli measurements, classical feedforward and discarding, chosen independently of the input and without a magic catalyst, whose accepted branches map $\psi^{\otimes 8}$ exactly to

$$|\mathrm{CCZ}\rangle = 2^{-3/2} \sum_{x\in\{0,1\}^3} (-1)^{x_1x_2x_3} |x\rangle; \tag{1}$$

acceptance may vanish on stabilizer inputs. With $p_\Lambda(\psi)$ the acceptance probability (the total weight of the accepted branches) of $\Lambda \in \mathcal{U}_8$ on $\psi^{\otimes 8}$ and $p_8^*(\psi) = \sup_{\Lambda\in\mathcal{U}_8} p_\Lambda(\psi)$, the record asks whether the candidate optimum

$$p_8^*(\psi) = \tfrac{2}{3}\, m_3(\psi) \tag{2}$$

holds for every pure qubit ψ. The class $\mathcal{U}_8$ consists of the eight-copy stabilizer protocols of [1], so Eq. (2) asserts that the value reached in [1] is optimal.

Theorem 1. *The protocol Λ^* of Definition 1 lies in $\mathcal{U}_8$. Each of its nine accepted branches outputs exactly $|\mathrm{CCZ}\rangle$ whenever it occurs, and for every pure qubit ψ with Bloch vector (x, y, z)*

$$p_{\Lambda^*}(\psi) = m_3(\psi) = \tfrac{3}{2}\,(1 - x^2)(1 - y^2)(1 - z^2). \tag{3}$$

Since $m_3 > 0$ away from the six stabilizer states, Eq. (2) fails at every non-stabilizer ψ, and $m_3 \le p_8^* \le \frac{7}{6} m_3$ (Corollary 7); the exact value of p_8^* remains open. The mechanism is as follows. After $X^{\otimes 8}$ and $Z^{\otimes 8}$ are measured, each nontrivial outcome pair selects a sector that meets the three-dimensional subspace M_8 of symmetric vectors orthogonal to all stabilizer product states in one direction E_P. Each accepted branch acts with rank one on the symmetric subspace, so its output does not depend on ψ, and a Clifford circuit decodes it to $|\mathrm{CCZ}\rangle$. Two branches per sector keep 4/7 of E_P, which gives $\frac{2}{3} m_3$; a third branch, on the outcome that these two discard, keeps another 2/7, and $\frac{6}{7} \cdot \frac{7}{6} m_3 = m_3$. The last seventh sits in a branch whose image in M_8 is a stabilizer state, which no continuation can use (Proposition 9).

2 Setting and notation

Conventions. Qubits are labelled $0, \ldots, 7$, and the basis state $|s\rangle$, $s \in \{0,1\}^8$, corresponds to the integer $\sum_q s_q 2^q$, so qubit q is bit q. We write $|s|$ for the Hamming weight, e_i for the string with a single 1 at position i, and $\bar{s} = s \oplus 1^8$. The Pauli matrices are X, Z and $Y = iXZ$. For $P \in \{X, Y, Z\}$ and $S \subseteq \{0, \ldots, 7\}$, P^S is P on every qubit of S; $P^{\otimes 8} = P^{\{0,\ldots,7\}}$ and $P_i P_j = P^{\{i,j\}}$. We use $E_1 = \{0, 1, 2, 3\}$ and $E_2 = \{0, 3, 4, 5\}$, and $\mathrm{CX}_{c,t}$ denotes the CNOT gate with control c and target t. The gates $H = (X + Z)/\sqrt{2}$ and $R = e^{-i\pi X/4} = (I - iX)/\sqrt{2} = e^{-i\pi/4} HSH$, with $S = \mathrm{diag}(1, i)$, are Clifford gates, and $HZH = X$, $RXR^\dagger = X$, $RZR^\dagger = -Y$. For a triple O of qubits, $|\mathrm{CCZ}\rangle_O$ is the state (1) on O; it is symmetric under permutations of O.

Symmetric subspace and M_8. For $\psi = a|0\rangle + b|1\rangle$, not necessarily normalised, let $u_w = \sum_{|s|=w} |s\rangle$ for $w = 0, \ldots, 8$. Then

$$\psi^{\otimes 8} = \sum_{w=0}^{8} a^{8-w} b^w\, u_w, \tag{4}$$

and $\mathrm{Sym}^8 = \mathrm{span}\{u_0, \ldots, u_8\}$, with orthogonal projector Π_{Sym}. For normalised ψ the Bloch vector is given by $x + iy = 2\bar{a}b$ and $z = |a|^2 - |b|^2$. The six stabilizer states τ are the multiples of $(a, b) = (1, 0), (0, 1), (1, \pm 1), (1, \pm i)$, the zeros of the sextic

$$f_6(a, b) = ab(a^4 - b^4) = ab(a - b)(a + b)(a - ib)(a + ib). \tag{5}$$

Following [1], M_8 is the set of $\Phi \in \mathrm{Sym}^8$ orthogonal to all six $\tau^{\otimes 8}$. For $\Phi = \sum_w c_w u_w$ the binary form $p_\Phi(\alpha, \beta) = \sum_w \binom{8}{w} c_w \alpha^{8-w} \beta^w$ satisfies $\langle \psi^{\otimes 8} | \Phi \rangle = p_\Phi(\bar{a}, \bar{b})$, and $\Phi \mapsto p_\Phi$ is an isomorphism onto the binary forms of degree 8. Complex conjugation permutes the six stabilizer points, so $\Phi \in M_8$ if and only if f_6 divides p_Φ. Hence $\{p_\Phi : \Phi \in M_8\} = f_6 \cdot \{\text{quadratic forms}\}$ and $\dim M_8 = 3$, as in [1, Lemma 4].

Sectors. The joint eigenspaces of the commuting operators $X^{\otimes 8}$, $Z^{\otimes 8}$ with eigenvalues $(-1, +1)$, $(+1, -1)$ and $(-1, -1)$ are called the sectors Z, X and Y. As $X^{\otimes 8} u_w = u_{8-w}$ and $Z^{\otimes 8} u_w = (-1)^w u_w$, the unit vector

$$E_Z = (u_2 - u_6)/\sqrt{56} \tag{6}$$

lies in sector Z, and $p_{E_Z} = \frac{28}{\sqrt{56}}\,\alpha\beta f_6(\alpha,\beta)$, so $E_Z \in M_8$. Let $E_X = H^{\otimes 8}E_Z$ and $E_Y = R^{\otimes 8}E_Z$. Conjugation by $H^{\otimes 8}$ exchanges $X^{\otimes 8}$ and $Z^{\otimes 8}$; conjugation by $R^{\otimes 8}$, in either direction, fixes $X^{\otimes 8}$ and maps $Z^{\otimes 8}$ to $Y^{\otimes 8} = X^{\otimes 8}Z^{\otimes 8}$. So E_X and E_Y lie in sectors X and Y. Both unitaries preserve Sym^8 and permute the $\tau^{\otimes 8}$ up to phases, hence preserve M_8. Thus (E_X, E_Y, E_Z) is an orthonormal basis of M_8 with one vector per sector, and the eigenspace $(+1,+1)$ is orthogonal to M_8.

Stabilizer operations. A Kraus operator of an accepted branch of $\Lambda \in \mathcal{U}_8$ is a composition of Clifford unitaries, tensoring with stabilizer states, Pauli projectors $(I \pm P)/2$ and maps $\langle s|_q$ that discard a qubit q. Each of these maps a stabilizer vector (a multiple of a stabilizer state) to a stabilizer vector or to zero [5]. Exactness means that $K\psi^{\otimes 8} \in \mathbb{C}|\mathrm{CCZ}\rangle$ for every such K and every ψ. The state $|\mathrm{CCZ}\rangle$ is not a stabilizer state: if $X^u Z^v|\mathrm{CCZ}\rangle \in \mathbb{C}|\mathrm{CCZ}\rangle$ then $x_1x_2x_3 + (x\oplus u)_1(x\oplus u)_2(x\oplus u)_3 + v\cdot x$ is constant modulo 2, which forces $u=0$ and then $v=0$, whereas a three-qubit stabilizer state is fixed by eight Pauli operators. Therefore $K\tau^{\otimes 8} = 0$ for every stabilizer state τ, and since Sym^8 is the orthogonal sum of M_8 and the span of the $\tau^{\otimes 8}$,

$$K\,\Pi_{\mathrm{Sym}} = K\,\Pi_{M_8} \qquad \text{for every accepted Kraus operator } K \text{ of every } \Lambda \in \mathcal{U}_8. \tag{7}$$

With $\sum_K K^\dagger K \le I$ this gives $p_\Lambda(\psi) \le \langle\psi^{\otimes 8}|\Pi_{M_8}|\psi^{\otimes 8}\rangle = \frac{7}{6}m_3(\psi)$ (Remark 8), which coincides with the bound of [1, Proposition 1] at eight copies.

3 The protocol

Definition 1 (the protocol Λ^*)**.** The input $\psi^{\otimes 8}$ occupies qubits $0,\dots,7$; no auxiliary qubits are used. (i) Measure $X^{\otimes 8}$, then $Z^{\otimes 8}$. Reject on $(+1,+1)$; otherwise let $P = Z$, X or Y for the outcomes $(-1,+1)$, $(+1,-1)$ or $(-1,-1)$. (ii) Measure P^{E_1}. On outcome -1, measure P_0P_1 and go to leaf A_1 (outcome $+1$) or A_2 (outcome -1). On outcome $+1$, measure P^{E_2}; go to leaf B on -1 and reject on $+1$. (iii) At leaf $L \in \{A_1, A_2, B\}$ of sector P, apply the Clifford unitary $D_L C_P^\dagger$, with $C_Z = I$, $C_X = H^{\otimes 8}$, $C_Y = R^{\otimes 8}$ and D_L from (8); output the qubits $O_{A_1} = O_{A_2} = (4,5,6)$, respectively $O_B = (0,1,4)$, and discard the other five.

The tree has 13 leaves: the rejected root outcome and, in each sector, A_1, A_2, B and the rejected leaf R ($P^{E_1} = P^{E_2} = +1$). With gates listed in the order in which they are applied, the decoders are

$$\begin{aligned}
D_{A_1} &= [\mathrm{CX}_{0,1}, \mathrm{CX}_{0,4}, \mathrm{CX}_{0,5}, \mathrm{CX}_{0,6}, \mathrm{CX}_{0,7}, H_0, X_0;\ \mathrm{CX}_{2,3}, H_2, X_3;\ G_{4,5,6,7}],\\
D_{A_2} &= [\mathrm{CX}_{2,3}, \mathrm{CX}_{2,4}, \mathrm{CX}_{2,5}, \mathrm{CX}_{2,6}, \mathrm{CX}_{2,7}, H_2, X_2;\ \mathrm{CX}_{0,1}, H_0, X_1;\ G_{4,5,6,7}],\\
D_B &= [\mathrm{CX}_{0,3}, \mathrm{CX}_{1,2}, \mathrm{CX}_{4,5}, \mathrm{CX}_{6,7}, \mathrm{CX}_{3,2}, \mathrm{CX}_{5,7}, \mathrm{CX}_{3,5}, X_5;\\
&\qquad H_0, H_1, H_4, H_6, \mathrm{CX}_{4,3}, \mathrm{CX}_{6,3};\ G_{0,1,4,6}],\\
G_{a,b,c,d} &= [\mathrm{CX}_{a,d}, \mathrm{CX}_{b,d}, \mathrm{CX}_{c,d}, X_d, \mathrm{CX}_{c,a}, \mathrm{CX}_{c,b}, H_c],
\end{aligned} \tag{8}$$

with 17, 17 and 21 gates. The leaf states of sector Z are

$$\chi_L = \sum_{\{i,j\}\in T_L} \left(|e_i + e_j\rangle - |\overline{e_i + e_j}\rangle\right), \qquad \begin{aligned} T_{A_1} &= \{2,3\} \times \{4,5,6,7\},\\ T_{A_2} &= \{0,1\} \times \{4,5,6,7\},\\ T_B &= \{0,3\} \times \{1,2\} \ \cup\ \{4,5\} \times \{6,7\}, \end{aligned} \tag{9}$$

where $I \times J$ denotes the pairs $\{i,j\}$ with $i \in I$ and $j \in J$. Each T_L contains 8 of the 28 pairs, so $\|\chi_L\|^2 = 16$. With $W_{ij} = |10\rangle_{ij} + |01\rangle_{ij}$, W_{abcd} the sum of the four weight-one strings on qubits a,b,c,d and $\overline{W}_{abcd}$ the sum of their complements,

$$\chi_{A_1} = W_{23}(|00\rangle_{01}W_{4567} - |11\rangle_{01}\overline{W}_{4567}), \qquad \chi_{A_2} = W_{01}(|00\rangle_{23}W_{4567} - |11\rangle_{23}\overline{W}_{4567}),$$
$$\chi_B = W_{03}W_{12}(|0000\rangle_{4567} - |1111\rangle_{4567}) + (|0000\rangle_{0123} - |1111\rangle_{0123})W_{45}W_{67}.$$

The leaf states of sectors X and Y are $H^{\otimes 8}\chi_L$ and $R^{\otimes 8}\chi_L$ (Lemma 3), which $D_L C_P^\dagger$ maps to $D_L\chi_L$.

4 Proof of the theorem

For a leaf L of sector P, Π_L^P denotes the product of the projectors $(I \pm Q)/2$ onto the observed outcomes of the Pauli operators Q measured on the path to L; in sector Z we write $\Pi_L = \Pi_L^Z$.

Lemma 2 (instrument and sector Z)**.** (a) *On each of the* 13 *paths the measured Pauli operators are Hermitian and commute pairwise, so the Kraus operator of the path before decoding is the orthogonal projector* Π_L^P*, and the* 13 *projectors sum to* I*.* (b) *For* $L \in \{A_1, A_2, B\}$*,* $\Pi_L u_w = 0$ *for* $w \notin \{2, 6\}$ *and* $\Pi_L u_2 = -\Pi_L u_6 = \chi_L/2$*. Hence*

$$\Pi_L\psi^{\otimes 8} = \tfrac{1}{2}\, ab\, f_6(a, b)\, \chi_L, \qquad \Pi_L\Pi_{\mathrm{Sym}} = \Pi_L|E_Z\rangle\langle E_Z|, \qquad \langle E_Z|\Pi_L|E_Z\rangle = \tfrac{16}{56} = \tfrac{2}{7}. \tag{10}$$

Proof. (a) $X^{\otimes 8}$ and $Z^{\otimes 8}$ commute with each other and with every P^S with $|S|$ even, Pauli operators of a single type commute, and E_1, E_2, $\{0, 1\}$ have even size. A product of commuting projectors is the projector onto the joint eigenspace, and the leaf Kraus operators K of any measurement tree satisfy $\sum K^\dagger K = I$. (b) In sector Z every condition except $X^{\otimes 8} = -1$ is diagonal. These conditions define a set S_L of strings, closed under complementation because all masks have even size, and $\Pi_L|s\rangle = \frac{1}{2}(|s\rangle - |\bar{s}\rangle)$ for $s \in S_L$, $\Pi_L|s\rangle = 0$ otherwise. Odd weights violate $Z^{\otimes 8} = +1$, and 0^8, 1^8 violate $Z^{E_1} = -1$ (A_1, A_2) or $Z^{E_2} = -1$ (B). Complementation permutes the weight-four strings of S_L, so $\Pi_L u_4 = 0$, and it maps the weight-two strings of S_L onto the weight-six ones, so $\Pi_L u_2 = -\Pi_L u_6 = \frac{1}{2}\sum(|e_i + e_j\rangle - |\overline{e_i + e_j}\rangle)$, summed over the pairs with $e_i + e_j \in S_L$. Exactly one of i, j lies in E_1 for 16 pairs, those with $Z^{E_1} = -1$; among them $Z_0Z_1 = +1$ selects T_{A_1} and $Z_0Z_1 = -1$ selects T_{A_2}. Of the other 12 pairs, $Z^{E_2} = -1$ holds for the 8 pairs of T_B and $Z^{E_2} = +1$ for $\{0, 3\}, \{1, 2\}, \{4, 5\}, \{6, 7\}$. By (4), $\Pi_L\psi^{\otimes 8} = \frac{1}{2}(a^6b^2 - a^2b^6)\chi_L = \frac{1}{2}abf_6\,\chi_L$. For $\Phi = \sum_w c_w u_w$, $\Pi_L\Phi = \frac{1}{2}(c_2 - c_6)\chi_L = \langle E_Z|\Phi\rangle\,\chi_L/\sqrt{56}$ and $\Pi_L E_Z = \chi_L/\sqrt{56}$. □

Lemma 3 (sectors X and Y)**.** *For each leaf label* L*,* $\Pi_L^X = H^{\otimes 8}\Pi_L H^{\otimes 8}$ *and* $\Pi_L^Y = R^{\otimes 8}\Pi_L R^{\dagger\otimes 8}$*. Hence, for* $L \in \{A_1, A_2, B\}$*,* $\Pi_L^X\psi^{\otimes 8} = \frac{1}{2}a'b'f_6(a', b')\, H^{\otimes 8}\chi_L$ *and* $\Pi_L^Y\psi^{\otimes 8} = \frac{1}{2}a''b''f_6(a'', b'')\, R^{\otimes 8}\chi_L$*, where* $(a', b') = H\psi = \frac{1}{\sqrt{2}}(a + b, a - b)$ *and* $(a'', b'') = R^\dagger\psi = \frac{1}{\sqrt{2}}(a + ib, ia + b)$*. Moreover* $\Pi_L^P\Pi_{\mathrm{Sym}} = \Pi_L^P|E_P\rangle\langle E_P|$ *and* $\langle E_P|\Pi_L^P|E_P\rangle = 2/7$ *for* $P \in \{X, Y\}$*.*

Proof. Conjugation by $H^{\otimes 8}$ maps Z^S to X^S and exchanges $X^{\otimes 8}$ and $Z^{\otimes 8}$, so it maps the projectors of the sector-Z path of L to those of the sector-X path. The map $Q \mapsto R^{\otimes 8}QR^{\dagger\otimes 8}$ fixes $X^{\otimes 8}$ and sends Z^S to $(-Y)^S = Y^S$ for $|S|$ even; where $X^{\otimes 8} = -1$, the image $Y^{\otimes 8} = X^{\otimes 8}Z^{\otimes 8}$ of $Z^{\otimes 8}$ equals $+1$ exactly where $Z^{\otimes 8} = -1$. So the sector-Z path of L goes to the sector-Y path. Now $\Pi_L^X\psi^{\otimes 8} = H^{\otimes 8}\Pi_L(H\psi)^{\otimes 8}$ and $\Pi_L^Y\psi^{\otimes 8} = R^{\otimes 8}\Pi_L(R^\dagger\psi)^{\otimes 8}$, and Lemma 2(b) applies. The last statements follow by the same conjugations, since $H^{\otimes 8}$ and $R^{\otimes 8}$ commute with Π_{Sym}. □

Lemma 4 (decoders)**.** *For* $L \in \{A_1, A_2, B\}$*,* $D_L\chi_L = 4\,|\mathrm{CCZ}\rangle_{O_L} \otimes |0\rangle^{\otimes 5}$*, with* $|0\rangle^{\otimes 5}$ *on the other qubits.*

Proof. Write $|+'\rangle = |0\rangle + |1\rangle$, so that $H|+'\rangle = \sqrt{2}|0\rangle$, $H(|0\rangle - |1\rangle) = \sqrt{2}|1\rangle$ and $H^{\otimes 2}(|00\rangle - |11\rangle) = |01\rangle + |10\rangle$. First, $G_{a,b,c,d}W_{abcd} = 2\,|\mathrm{CCZ}\rangle_{(a,b,c)}|0\rangle_d$: the three CX gates into d followed by X_d map W_{abcd} to $(|000\rangle + |100\rangle + |010\rangle + |001\rangle)_{abc}|0\rangle_d$; the gates $\mathrm{CX}_{c,a}$ and $\mathrm{CX}_{c,b}$ map this to $\sum_{u,v\in\{0,1\}}|u, v, uv\rangle_{abc}|0\rangle_d$; and H_c gives $2^{-1/2}\sum_{u,v,t}(-1)^{uvt}|u, v, t\rangle_{abc}|0\rangle_d = 2\,|\mathrm{CCZ}\rangle_{(a,b,c)}|0\rangle_d$.

D_{A_1}: the gates $\mathrm{CX}_{0,1}$ and $\mathrm{CX}_{0,4}, \ldots, \mathrm{CX}_{0,7}$ map $|11\rangle_{01}\overline{W}_{4567}$ to $|10\rangle_{01}W_{4567}$ and fix $|00\rangle_{01}W_{4567}$, so $\chi_{A_1} \mapsto W_{23}(|0\rangle - |1\rangle)_0|0\rangle_1 W_{4567}$; then H_0 and X_0 turn $(|0\rangle - |1\rangle)_0$ into $\sqrt{2}|0\rangle_0$. The gate $\mathrm{CX}_{2,3}$ maps W_{23} to $|+'\rangle_2|1\rangle_3$, and H_2, X_3 give $\sqrt{2}|00\rangle_{23}$. The result is $2|0000\rangle_{0123}W_{4567}$, and $G_{4,5,6,7}$ completes the claim. D_{A_2} is identical with the pairs $(0, 1)$ and $(2, 3)$ exchanged.

D_B: on each pair $(i,j) \in \{(0,3),(1,2),(4,5),(6,7)\}$ the gate $\mathrm{CX}_{i,j}$ maps $W_{ij} \mapsto |{+'}\rangle_i|1\rangle_j$, $|00\rangle \mapsto |00\rangle$ and $|11\rangle \mapsto |10\rangle$. The gates $\mathrm{CX}_{3,2}, \mathrm{CX}_{5,7}, \mathrm{CX}_{3,5}, X_5$ then reset qubits 2, 5 and 7 to $|0\rangle$, and the state on qubits $0,1,3,4,6$ becomes

$$|1\rangle_3|{+'}\rangle_0|{+'}\rangle_1(|00\rangle - |11\rangle)_{46} + |0\rangle_3(|00\rangle - |11\rangle)_{01}|{+'}\rangle_4|{+'}\rangle_6.$$

The Hadamard gates on qubits $0,1,4,6$ turn this into $2|1\rangle_3|00\rangle_{01}(|01\rangle + |10\rangle)_{46} + 2|0\rangle_3(|01\rangle + |10\rangle)_{01}|00\rangle_{46}$. The gates $\mathrm{CX}_{4,3}$ and $\mathrm{CX}_{6,3}$ reset qubit 3, since exactly one of qubits $4,6$ is set in the first term and neither in the second. This leaves $2W_{0146}$, and $G_{0,1,4,6}$ completes the claim. □

Lemma 5 (acceptance probabilities). *Let $m = (a^{8-w}b^w)_{w=0}^8$ and $v_k = \hat{e}_k - \hat{e}_{k+4}$ for $k = 1,2,3$, where $\hat{e}_0, \ldots, \hat{e}_8$ are the unit vectors of $\mathbb{R}^9$. For an accepted leaf L of sector P the matrix $N^P_{ww'} = \langle u_w|\Pi^P_L|u_{w'}\rangle$ is $N^Z = 4v_2v_2^\mathsf{T}$, $N^X = (v_1 - v_3)(v_1 - v_3)^\mathsf{T}$ or $N^Y = (v_1 + v_3)(v_1 + v_3)^\mathsf{T}$, and $\|\Pi^P_L\psi^{\otimes 8}\|^2 = m^\dagger N^P m$. Summed over the nine accepted leaves,*

$$N = 6\big(v_1v_1^\mathsf{T} + 2v_2v_2^\mathsf{T} + v_3v_3^\mathsf{T}\big), \qquad \sum_{P,L} \|\Pi^P_L\psi^{\otimes 8}\|^2 = m^\dagger N m = 6\,|f_6(a,b)|^2\big(|a|^2 + |b|^2\big)^2. \tag{11}$$

Proof. From $v_k^\mathsf{T} m = a^{4-k}b^k(a^4 - b^4)$ one gets $v_1^\mathsf{T} m = a^2 f_6$, $v_2^\mathsf{T} m = ab f_6$ and $v_3^\mathsf{T} m = b^2 f_6$. By Lemma 2, $\|\Pi_L\psi^{\otimes 8}\|^2 = \frac{1}{4}|abf_6|^2\|\chi_L\|^2 = 4|v_2^\mathsf{T} m|^2$. In sector X, $a'b' = \frac{1}{2}(a^2 - b^2)$, $a'^2 - b'^2 = 2ab$ and $a'^2 + b'^2 = a^2 + b^2$, so $f_6(a',b') = f_6(a,b)$ and, by Lemma 3, $\|\Pi^X_L\psi^{\otimes 8}\|^2 = |(a^2 - b^2)f_6|^2 = |(v_1 - v_3)^\mathsf{T} m|^2$. In sector Y, $a''b'' = \frac{i}{2}(a^2 + b^2)$, $a''^2 - b''^2 = a^2 - b^2$ and $a''^2 + b''^2 = 2iab$, so $f_6(a'',b'') = -f_6(a,b)$ and $\|\Pi^Y_L\psi^{\otimes 8}\|^2 = |(a^2 + b^2)f_6|^2 = |(v_1 + v_3)^\mathsf{T} m|^2$. By (4), $\|\Pi^P_L\psi^{\otimes 8}\|^2 = \sum_{w,w'} \bar{m}_w N^P_{ww'} m_{w'}$; since the functions $(a,b) \mapsto \bar{a}^{8-w}\bar{b}^w a^{8-w'}b^{w'}$ on $\mathbb{C}^2$ are linearly independent, this Hermitian form determines N^P. Three leaves per sector give $N = 3\big[4v_2v_2^\mathsf{T} + (v_1 - v_3)(v_1 - v_3)^\mathsf{T} + (v_1 + v_3)(v_1 + v_3)^\mathsf{T}\big]$, and the values of $v_k^\mathsf{T} m$ give $m^\dagger N m$. □

Lemma 6 (Bloch form). *For $(a,b) \in \mathbb{C}^2$ let $x + iy = 2\bar{a}b$, $z = |a|^2 - |b|^2$ and $n = |a|^2 + |b|^2$. Then $x^2 + y^2 + z^2 = n^2$ and*

$$\tfrac{1}{2}(n^6 - x^6 - y^6 - z^6) = 6\,|f_6(a,b)|^2 = \tfrac{3}{2}\,(n^2 - x^2)(n^2 - y^2)(n^2 - z^2).$$

In particular $m_3(\psi) = 6|f_6|^2 = \frac{3}{2}(1 - x^2)(1 - y^2)(1 - z^2)$ for normalised ψ.

Proof. First, $x^2 + y^2 + z^2 = 4|a|^2|b|^2 + (|a|^2 - |b|^2)^2 = n^2$ and $4|ab|^2 = n^2 - z^2$. Next, $|a^2 \pm b^2|^2 = |a|^4 + |b|^4 \pm 2\,\mathrm{Re}\big[(\bar{a}b)^2\big] = \frac{1}{2}(n^2 + z^2) \pm \frac{1}{2}(x^2 - y^2)$, so $|a^2 - b^2|^2 = n^2 - x^2$ and $|a^2 + b^2|^2 = n^2 - y^2$. As $|f_6|^2 = |ab|^2|a^2 - b^2|^2|a^2 + b^2|^2$, this gives $4|f_6|^2 = (n^2 - x^2)(n^2 - y^2)(n^2 - z^2)$. Finally $(\xi + \eta + \zeta)^3 - \xi^3 - \eta^3 - \zeta^3 = 3(\xi + \eta)(\eta + \zeta)(\zeta + \xi)$, applied to $\xi = x^2$, $\eta = y^2$, $\zeta = z^2$ with $\xi + \eta + \zeta = n^2$, gives the first equality. □

Proof of Theorem 1. The protocol Λ^* uses only measurements of Pauli operators, choices of later measurements and gates that depend on earlier outcomes (classical feedforward), the Clifford gates CX, X, H and $R = e^{-i\pi/4}HSH$, and discarding. It uses no auxiliary state and no catalyst, and nothing in it depends on ψ. At an accepted leaf L of sector P, Lemmas 2 and 3 give $\Pi^P_L\psi^{\otimes 8} = c\,C_P\chi_L$ with a scalar $c = c_{P,L}(\psi)$, so by Lemma 4 $D_L C_P^\dagger \Pi^P_L\psi^{\otimes 8} = 4c\,|\mathrm{CCZ}\rangle_{O_L} \otimes |0\rangle^{\otimes 5}$. Whenever the leaf occurs ($c \neq 0$), discarding the five qubits in $|0\rangle^{\otimes 5}$ leaves exactly $|\mathrm{CCZ}\rangle_{O_L}$; hence $\Lambda^* \in \mathcal{U}_8$. By Lemma 2(a) the leaf occurs with probability $\|\Pi^P_L\psi^{\otimes 8}\|^2$, and Lemmas 5 and 6 give $p_{\Lambda^*}(\psi) = 6|f_6|^2 = m_3(\psi)$ for normalised ψ. □

5 Consequences

Corollary 7. *For every pure qubit ψ, $m_3(\psi) \le p_8^*(\psi) \le \frac{7}{6}m_3(\psi)$, and $m_3(\psi) = 0$ exactly at the six stabilizer states. Hence Eq. (2) fails at every non-stabilizer ψ, where $p_8^* \ge \frac{3}{2} \cdot \frac{2}{3}m_3$, and holds,* *with both sides zero, at the six stabilizer states. At the T-type state, with Bloch vector* $(1,1,1)/\sqrt{3}$, $p_{\Lambda^*} = m_3 = 4/9$, *compared with* $\frac{2}{3}m_3 = 8/27$ *and* $\frac{7}{6}m_3 = 14/27$*; at the H-type state, with Bloch vector* $(1,0,1)/\sqrt{2}$ *[3], the three values are* $3/8$, $1/4$ *and* $7/16$.

Proof. The lower bound is Theorem 1, and the upper bound is [1, Proposition 1] (see also (7)). By Lemma 6, $m_3 = 0$ exactly when a Bloch component has modulus 1, that is, at the six stabilizer states. The values follow from $m_3 = \frac{3}{2}(1-x^2)(1-y^2)(1-z^2)$. □

Remark 8 (mechanism). By Lemmas 2 and 3, $\Pi_{\mathrm{Sym}}\Pi_L^P\Pi_{\mathrm{Sym}} = \frac{2}{7}|E_P\rangle\langle E_P|$ for each accepted leaf: the leaf keeps $2/7$ of E_P, the ratio of its 8 pairs to all 28 pairs. Summing over the nine leaves, $\sum_{P,L}\Pi_{\mathrm{Sym}}\Pi_L^P\Pi_{\mathrm{Sym}} = \frac{6}{7}\Pi_{M_8}$. By Lemma 5, the matrix of Π_{M_8} in the basis (u_w) is therefore $\frac{7}{6}N = 7(v_1v_1^\top + 2v_2v_2^\top + v_3v_3^\top)$, and $\langle\psi^{\otimes 8}|\Pi_{M_8}|\psi^{\otimes 8}\rangle = 7|f_6|^2 = \frac{7}{6}m_3$. The six leaves A_1, A_2 keep $4/7$ of each E_P and give $4(v_1v_1^\top + 2v_2v_2^\top + v_3v_3^\top)$, that is, $\frac{4}{7}\cdot\frac{7}{6}m_3 = \frac{2}{3}m_3$. This is the success probability of the eight-copy protocol of [1], which also keeps $4/7$ of the weight in M_8; we do not claim that the leaves A_1, A_2 coincide gate by gate with that protocol. The leaf B acts on the outcome $+P^{E_1}$, which the leaves A_1, A_2 discard, and adds $2/7$ per sector.

Proposition 9 (the missing seventh)**.** *In sector Z the rejected leaf R satisfies* $\Pi_R\psi^{\otimes 8} = \frac{1}{2}(a^8 - b^8)\,\mathrm{GHZ}^- + \frac{1}{2}\,ab\,f_6\,\chi_C$*, where* $\mathrm{GHZ}^- = |0^8\rangle - |1^8\rangle$ *and* $\chi_C = \sum(|e_i+e_j\rangle - |\overline{e_i+e_j}\rangle)$*, summed over the pairs* $\{0,3\},\{1,2\},\{4,5\},\{6,7\}$*. Both vectors are stabilizer vectors,* Π_R *has rank two on* Sym^8*, and* $\langle E_Z|\Pi_R|E_Z\rangle = 1/7$*. Let* $\Lambda \in \mathcal{U}_8$ *first perform the measurements of* Λ^* *and then continue in any way. Every accepted branch of* Λ *through the root outcome* $(+1,+1)$ *or through a leaf R has probability zero for every* ψ*, and* $p_\Lambda \le p_{\Lambda^*} = m_3$.

Proof. The formula follows as in Lemma 2(b); now $0^8, 1^8 \in S_R$, and the four listed pairs are those with $e_i + e_j \in S_R$. The vector GHZ^- is a stabilizer vector, and the Clifford circuit $[\mathrm{CX}_{0,3}, \mathrm{CX}_{1,2}, \mathrm{CX}_{4,5}, \mathrm{CX}_{6,7}, \mathrm{CX}_{0,6}, \mathrm{CX}_{1,6}, \mathrm{CX}_{4,6}, X_6]$ maps χ_C to the triangle graph state $\sum_{u\in\{0,1\}^3}(-1)^{u_1u_2+u_1u_3+u_2u_3}|u\rangle_{0,1,4}$ times $|0\rangle^{\otimes 5}$. The capture is $\|\Pi_R E_Z\|^2 = \|\chi_C\|^2/56 = 1/7$. An accepted Kraus operator of Λ through R has the form $K = K'\Pi_R$, with K' a composition of stabilizer operations. By (7) and $\Pi_R E_X = \Pi_R E_Y = 0$, $K\psi^{\otimes 8} = K'\Pi_R\Pi_{M_8}\psi^{\otimes 8} = \langle E_Z|\psi^{\otimes 8}\rangle\,K'\chi_C/\sqrt{56}$. Exactness at any ψ with $\langle E_Z|\psi^{\otimes 8}\rangle \ne 0$ forces $K'\chi_C \in \mathbb{C}|\mathrm{CCZ}\rangle$; as $K'\chi_C$ is a stabilizer vector or zero, $K'\chi_C = 0$. Sectors X and Y follow by the conjugations of Lemma 3, and the root outcome $(+1,+1)$ because its projector annihilates M_8. From an accepted leaf of Λ^* no continuation accepts with more than the probability of the leaf; summing over the leaves gives $p_\Lambda \le m_3$. □

Measuring $Z^{\{0,1,4,6\}}$ on the leaf R of sector Z isolates the missing seventh (outcome -1 gives a leaf of rank one on Sym^8 with capture $1/7$), but its state χ_C is a stabilizer state. Reaching $\frac{7}{6}m_3$ requires the accepted branches to keep all of M_8, which by Proposition 9 needs a measurement tree that does not refine the one of Λ^*. Conversely, an upper bound below $\frac{7}{6}m_3$ must use more of the stabilizer structure than condition (7), which alone permits $\frac{7}{6}m_3$. The exact value of p_8^* in $[m_3, \frac{7}{6}m_3]$ remains open.

6 Verification

The file `check_eight_copy_ccz.py` is standalone. It uses only the Python standard library, exact arithmetic with integers, Gaussian integers and fractions, and no network, and it takes no input; its output is fixed in `expected_stdout.txt` (20 `PASS` lines and a verdict). It builds the 13-leaf tree from Pauli masks and applies each path as a product of factors $I \pm Q$, so commutation is checked rather than assumed. It checks its Pauli and gate conventions; commutation on every path and $\sum \Pi_L^P = I$ on all 256 basis vectors; the rank-one property and the states (9); the captures $2/7$; the decoders (8) gate by gate, also on sector-X and sector-Y leaf states computed from their own projectors; the Pauli spectrum of all leaf states; Eq. (11), $\dim M_8 = 3$ and $N = \frac{6}{7}$ times the matrix of Π_{M_8}; Lemma 6 and $m^\dagger N m = 6|f_6|^2|\psi|^4$ as polynomial identities in $a, b, \bar{a}, \bar{b}$; and $p_{\Lambda^*} = m_3$ at three Gaussian-rational inputs. As controls, the leaves A_1, A_2 give $\frac{2}{3}m_3$, wrong decoders fail, the leaf isolating χ_C outputs a stabilizer state, and R has rank two. The program prints 21 lines in under a second, byte-identically on the host and in an isolated container without network (`reproduction.json`). The harness `mutate.py` applies 25 single-point mutations (masks, outcome signs, sectors, decoder gates, conventions and coefficients), each of which makes the program exit with a nonzero status, and six equivalent edits, such as $E_2 \to E_1 \oplus E_2$, which leave its output unchanged (`MUTATION_REPORT.txt`). Reproduction shows only that the program computes what it prints. Theorem 1 rests on the written argument of Section 4, and the program checks every finite identity that the argument uses.

AI assistance and review status

AI tools were used to develop the argument, to write the verification program, to assist with exposition and to conduct internal adversarial reviews. The manuscript follows the evidence-tracking guidance of Scientific Agent Skills [6]. These activities are disclosed as AI assistance, not as external refereeing or formal proof verification.

The protocol was found in an internal research round, after an earlier round had reproduced the bound $\frac{7}{6}m_3$. An adversarial review was then performed by AI in an isolated context, from the claim text alone. It rebuilt the tree with sequential products of projectors and verified exactly the completeness of the instrument, the rank-one property and the states of all nine accepted leaves, the decoders (with the sector-X and sector-Y leaves built from their own projectors), Eq. (11), the Bloch conversion and membership in the class of stabilizer protocols of [1]; a floating-point run over 60 random inputs agreed to within 3.3×10^{-15}. A separate simulation, also written from the claim text alone, decoded every accepted branch for 40 random inputs with fidelity $1 - 4 \times 10^{-15}$ and reproduced $p_{\Lambda^*} = m_3$ to within 1.3×10^{-15}. **No independent human expert has examined this argument, and no novelty certification by a human is recorded.** On 28 September 2026 the preprint [1] had a single version (v1, 13 August 2026); its full text was read by AI and states that optimality at eight copies remains open. Two web searches and arXiv checks on that day found no later work on eight-copy universal concentration, and the problem record [2] was still listed as unsolved.

References

[1] J. Rizzo and L. Leone, "Universal magic state concentration," arXiv:2608.13376v1 (2026). Theorem, proposition and equation numbers follow the arXiv HTML rendering of v1.

[2] Quantum Information and Quantum Computation Open Problem Zoo, "Optimal universal eight-copy concentration to a CCZ state," https://qiqc-op.com/problem/op_bc61eaba5c332fec/. Listed as unsolved on 28 September 2026.

[3] S. Bravyi and A. Kitaev, "Universal quantum computation with ideal Clifford gates and noisy ancillas," Phys. Rev. A **71**, 022316 (2005), doi:10.1103/PhysRevA.71.022316.

[4] L. Leone, S. F. E. Oliviero and A. Hamma, "Stabilizer Rényi entropy," Phys. Rev. Lett. **128**, 050402 (2022), arXiv:2106.12587.

[5] S. Aaronson and D. Gottesman, "Improved simulation of stabilizer circuits," Phys. Rev. A **70**, 052328 (2004), arXiv:quant-ph/0406196.

[6] T. Kassis, V. Agarwal, Y. He, D. Patel and A. M. Brueckner, "Scientific Agent Skills: A Library of Procedural Knowledge for Research Agents," arXiv:2609.00065v2 (2026). Preprint; Software repository. The scientific-writing guidance was consulted at commit `330c8e764435`.