Additivity of minimum output Rényi entropy can fail first at three copies

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Abstract

The minimum output Rényi-p entropy of a quantum channel is not additive in general, and the constructions we are aware of exhibit the violation already for two channel uses. Ruskai asked whether additivity can hold for all tensor powers below some order m and fail first at the mth. We answer this affirmatively for $m = 3$. We construct an explicit finite-dimensional channel Φ with $S_{p,\min}(\Phi^{\otimes 2}) = 2\,S_{p,\min}(\Phi)$ and $S_{p,\min}(\Phi^{\otimes 3}) < 3\,S_{p,\min}(\Phi)$ at $p = 1/1000$; channels of the same form, with a p-dependent constant block, do the same at every sufficiently small $p > 0$. The channel is a flagged direct sum of a constant channel with a channel Ψ from $\mathbb{C}^4$ to $\mathbb{C}^3$ whose six Kraus operators are supported on disjoint pairs of matrix entries. The mechanism is the minimum output rank. Every output of $\Psi \otimes \Psi$ has full rank $9 = 3^2$, and an input with a three-party GHZ structure gives $\Psi^{\otimes 3}$ an output of rank at most $26 < 27$. The two-copy statement is certified by an exact decomposable block-positivity certificate over $\mathbb{Z}[\omega]$, $\omega = e^{2\pi i/3}$. A standalone program verifies every finite computation exactly, using only the Python standard library.

1 Introduction

For a density operator σ and $p > 0$, $p \neq 1$, the Rényi entropy is $S_p(\sigma) = \frac{1}{1-p} \log_2 \operatorname{Tr} \sigma^p$, and for a channel Φ the minimum output entropy is $S_{p,\min}(\Phi) = \min_\rho S_p(\Phi(\rho))$ over input states. Product inputs give $S_{p,\min}(\Phi^{\otimes n}) \leq n\,S_{p,\min}(\Phi)$. Additivity of this quantity was conjectured and then refuted. Hastings found violations at $p = 1$ using random channels [2]. Cubitt, Harrow, Leung, Montanaro and Winter proved by a random construction that violations exist at $p = 0$ and hence for sufficiently small p, and gave an explicit pair of channels from 4 to 3 dimensions whose minimum output rank is not multiplicative [3]. Derksen and Lovitz gave explicit channels violating additivity for all $p > 1$ [4]. In all of these, and in recent explicit constructions [5], the violation is exhibited for two channel uses.

Ruskai asked whether additivity can hold for all tensor powers below an order m and first fail at the mth power [1, Problem 19]. The question is recorded as an open problem in [7], which asks for $p > 0$, an integer $m \geq 3$ and a channel Φ with

$$S_{p,\min}(\Phi^{\otimes n}) = n\,S_{p,\min}(\Phi) \quad (1 \leq n < m), \qquad S_{p,\min}(\Phi^{\otimes m}) < m\,S_{p,\min}(\Phi). \tag{1}$$

The restriction $m \geq 3$ makes the lower-power requirement nontrivial.

Theorem 1. *Let Ψ be the channel of Definition 4 and $s_k := S_{p,\min}(\Psi^{\otimes k})$. For a state τ on $\mathbb{C}^3$ let $\Phi_\tau : \mathcal{L}(\mathbb{C}^4 \oplus \mathbb{C}) \to \mathcal{L}(\mathbb{C}^3 \otimes \mathbb{C}^2)$ be the flagged channel*

$$\Phi_\tau(\rho) = \Psi(\Pi\rho\Pi) \otimes |0\rangle\langle 0| + \langle e|\rho|e\rangle\,\tau \otimes |1\rangle\langle 1|,$$

with Π the projector onto $\mathbb{C}^4$ and e the added basis vector. (a) *At $p = 1/1000$, the explicit choice $\tau_\star = \mathrm{diag}(1 - 2^{-20}, 2^{-21}, 2^{-21})$ gives $S_{p,\min}(\Phi_{\tau_\star}) = S_p(\tau_\star)$, $S_{p,\min}(\Phi_{\tau_\star}^{\otimes 2}) = 2S_p(\tau_\star)$ and $S_{p,\min}(\Phi_{\tau_\star}^{\otimes 3}) = s_3 < 3S_p(\tau_\star)$, so $\Phi_{\tau_\star}$ satisfies (1) with $m = 3$.* (b) *For every sufficiently small $p > 0$ there is a state τ_p for which Φ_{τ_p} satisfies (1) with $m = 3$; any τ with $s_3/3 < S_p(\tau) \le s_2/2$ works.*

The proof has three ingredients. A flagged direct sum turns the question into a statement about Ψ alone (Section 2): it suffices that $s_3 < \frac{3}{2}s_2$. For small p the entropies are governed by ranks. Every output of $\Psi \otimes \Psi$ has full rank 9, with an explicit lower bound on its smallest eigenvalue (Section 4). A particular input makes an output of $\Psi^{\otimes 3}$ lose one dimension (Section 5). Section 6 turns these rank facts into entropy inequalities at $p = 1/1000$. The two-copy bound and the numerical inequalities at $p = 1/1000$ are the only steps that need computation. Both are certified exactly, the first in two independent ways (Section 7).

2 Flagged direct sums

Throughout, $0 < p < 1$, logarithms are base 2, $s_0 := 0$, and $s_k := S_{p,\min}(\Psi^{\otimes k})$ for a channel Ψ.

Lemma 2. *Let Ψ be a channel, τ a state with $S_p(\tau) = b$, and $\Phi = \Phi_\tau$ as in Theorem 1 (for any input and output dimensions of Ψ). Then for every $n \ge 1$,*

$$S_{p,\min}(\Phi^{\otimes n}) = \min_{0 \le k \le n} \big[s_k + (n-k)\, b \big].$$

Proof. Write $\Phi = \sum_{f \in \{0,1\}} \Phi_f \otimes |f\rangle\langle f|$ with $\Phi_0 = \Psi(\Pi \cdot \Pi)$ and $\Phi_1 = \langle e| \cdot |e\rangle\, \tau$. Then $\Phi^{\otimes n} = \sum_{x \in \{0,1\}^n} (\bigotimes_i \Phi_{x_i}) \otimes |x\rangle\langle x|$, so for every input ρ the output is block diagonal over flag strings, $\bigoplus_x w_x \sigma_x$ with $w_x \ge 0$, $\sum_x w_x = 1$, and $\sigma_x = \Psi^{\otimes k}(\omega_x) \otimes \tau^{\otimes(n-k)}$ (up to ordering of the factors) for a state ω_x and k the number of zeros of x. Hence $S_p(\sigma_x) \ge s_k + (n-k)b$. Since $w \mapsto w^p$ is concave with $w^p \ge w$ on $[0,1]$,

$$\mathrm{Tr}\Big(\bigoplus_x w_x \sigma_x\Big)^p = \sum_x w_x^p \,\mathrm{Tr}\,\sigma_x^p \;\ge\; \Big(\min_x \mathrm{Tr}\,\sigma_x^p\Big) \sum_x w_x^p \;\ge\; \min_x \mathrm{Tr}\,\sigma_x^p,$$

so the output entropy is at least $\min_x S_p(\sigma_x)$. Inputs supported on a single flag block attain every value $s_k + (n-k)b$. □

Lemma 3. *Let $0 < p < 1$ and let τ satisfy $s_3/3 < b := S_p(\tau) \le s_2/2$. Then $S_{p,\min}(\Phi_\tau) = b$, $S_{p,\min}(\Phi_\tau^{\otimes 2}) = 2b$ and $S_{p,\min}(\Phi_\tau^{\otimes 3}) = s_3 < 3b$.*

Proof. Product inputs give $s_2 \le 2s_1$, so $b \le s_2/2 \le s_1$. By Lemma 2: $n = 1$ gives $\min(b, s_1) = b$; $n = 2$ gives $\min(2b,\ s_1 + b,\ s_2) = 2b$ because $2b \le s_2$; and $n = 3$ gives $\min(3b,\ s_1 + 2b,\ s_2 + b,\ s_3) = \min(3b, s_3) = s_3$, because $s_1 + 2b \ge 3b$, $s_2 + b \ge 3b$ and $s_3 < 3b$. □

The window $(s_3/3,\ s_2/2]$ is nonempty exactly when $s_3 < \frac{3}{2}s_2$, and then a state τ with $S_p(\tau)$ in it exists, because S_p is continuous on the connected state space of $\mathbb{C}^3$ and takes every value in $[0, \log 3]$, while $s_2/2 \le s_1 \le \log 3$.

3 The channel

Definition 4. Let $\omega = e^{2\pi i/3}$. For the six pairs of cells $((r_1,k_1),(r_2,k_2))$ of the 3×4 grid

$$((0,0),(2,2)),\ ((0,1),(1,2)),\ ((0,2),(2,1)),\ ((0,3),(1,0)),\ ((1,1),(2,3)),\ ((1,3),(2,0))$$

and $\theta = (1,1,1,1,1,\omega)$, put $K_j = \frac{1}{\sqrt{3}}(|r_1\rangle\langle k_1| + \theta_j |r_2\rangle\langle k_2|)$ and $\Psi(X) = \sum_{j=1}^6 K_j X K_j^\dagger$, a map from $\mathcal{L}(\mathbb{C}^4)$ to $\mathcal{L}(\mathbb{C}^3)$.

The six pairs partition the twelve cells. In each pair the two cells lie in different rows and different columns, and one column lies in $\{0,1\}$ and the other in $\{2,3\}$. Because the two cells of a pair lie in different rows, $K_j^\dagger K_j = \frac{1}{3}(|k_1\rangle\langle k_1| + |k_2\rangle\langle k_2|)$, and summing over the partition gives $\sum_j K_j^\dagger K_j = I_4$: Ψ is a channel. We write $\tilde{K}_j = \sqrt{3}K_j$, matrices with entries in $\{0,1,\omega\}$.

4 Two copies: full rank

For $y \in \mathbb{C}^9$ and $v \in \mathbb{C}^{16}$ put $h(y,v) = \sum_{a,b} |y^\dagger(\tilde{K}_a \otimes \tilde{K}_b)v|^2$. For a state $\rho = \sum_i \ell_i |v_i\rangle\langle v_i|$ and a unit vector y, $y^\dagger(\Psi\otimes\Psi)(\rho)\, y = \frac{1}{9}\sum_i \ell_i h(y, v_i)$. Writing $y^\dagger(\tilde{K}_a \otimes \tilde{K}_b)v = N_{ab}\cdot x$ with $x = \bar{y}\otimes v \in \mathbb{C}^9 \otimes \mathbb{C}^{16}$ and $N_{ab}[(s,s'),(k,k')] = \tilde{K}_a[s,k]\tilde{K}_b[s',k']$ gives $h(y,v) = x^\dagger W x$, where $W = N^\dagger N$ is a 144×144 matrix over $\mathbb{Z}[\omega]$.

Proposition 5. *$x^\dagger W x \geq 2^{-14}|x|^2$ for every product vector $x = u \otimes v$. Consequently $\lambda_{\min}\big((\Psi \otimes \Psi)(\rho)\big) \geq 2^{-14}/9$ for every state ρ on $\mathbb{C}^{16}$, every output of $\Psi \otimes \Psi$ has rank 9, and every output of Ψ has rank 3.*

Proof. Let $c_0 = 2^{-14}$. The program exhibits $R \in (2^{-32}\mathbb{Z}[\omega])^{144\times 110}$ and, with $Q = RR^\dagger \succeq 0$ and Q^Γ its partial transpose on the 16-dimensional factor,

$$P := W - c_0 I - Q^\Gamma \;\succ\; 0.$$

Positive definiteness of P is proved exactly in two independent ways (Section 7). For a product vector $x = u \otimes v$, $x^\dagger Q^\Gamma x = (u \otimes \bar{v})^\dagger Q\,(u \otimes \bar{v}) \geq 0$, so $x^\dagger W x = x^\dagger P x + x^\dagger Q^\Gamma x + c_0|x|^2 \geq c_0|x|^2$. Hence $h(y,v) \geq c_0$ for unit y, v, and $\lambda_{\min}((\Psi\otimes\Psi)(\rho)) \geq c_0/9$ for every state. If some output of Ψ had rank at most 2, then $(\Psi\otimes\Psi)(\rho\otimes\rho)$ would have rank at most 4. □

Remark 6. The certificate was found by a semidefinite program (the first level of the Doherty–Parrilo–Spedalieri hierarchy) and rounded to exact values; the program never trusts the solver. The certified constant is not tight: the relaxation optimum is about $2.1 \cdot 10^{-4}$, and a numerical search over product vectors finds a minimum of $x^\dagger W x$ near $9.1 \cdot 10^{-4}$. The construction is sensitive to θ: with all $\theta_j = 1$, the vectors $y \in \mathbb{C}^3 \otimes \mathbb{C}^3$ and $v \in \mathbb{C}^4 \otimes \mathbb{C}^4$ whose coefficient matrices are I_3 and $\mathrm{diag}(1,1,-1,-1)$ give $h(y,v) = 0$ exactly, so that channel's two-copy outputs lose rank.

5 Three copies: a rank drop

Proposition 7. *For $\varphi \propto |000\rangle + |111\rangle - |222\rangle - |333\rangle \in (\mathbb{C}^4)^{\otimes 3}$ and $\psi = \sum_{k=0}^{2} |kkk\rangle \in (\mathbb{C}^3)^{\otimes 3}$, $\psi^\dagger(\tilde{K}_a \otimes \tilde{K}_b \otimes \tilde{K}_c)\varphi = 0$ for all a, b, c. Hence $\operatorname{rank} \Psi^{\otimes 3}(|\varphi\rangle\langle\varphi|) \leq 26$.*

Proof. With $d = (1,1,-1,-1)$ the quantity is $\sum_{r,k} \tilde{K}_a[r,k]\tilde{K}_b[r,k]\tilde{K}_c[r,k]\, d_k$. A cell lies in exactly one pair, so only $a = b = c = j$ contributes, giving $d_{k_1} + \theta_j^3 d_{k_2} = d_{k_1} + d_{k_2} = 0$: $\theta_j^3 = 1$, and each pair joins a column in $\{0,1\}$ to one in $\{2,3\}$. So ψ is orthogonal to the range of the three-copy output. □

6 Entropies at small p

Lemma 8. *Let $\mu = 2^{-14}/9$ and $B(p) = \frac{1}{1-p}\log\big(8\mu^p + (1-8\mu)^p\big)$. For $0 < p < 1$, $s_3 \leq \log 26$ and $s_2 \geq B(p)$. Moreover $\frac{3}{2}B(1/1000) > \log 26$, and $\frac{3}{2}B(p) > \log 26$ for all sufficiently small $p > 0$.*

Proof. For a state of rank r, the power-mean inequality gives $\operatorname{Tr}\sigma^p \leq r^{1-p}$, so $S_p(\sigma) \leq \log r$; with Proposition 7, $s_3 \leq \log 26$. Every two-copy output has spectrum in $\{\lambda : \lambda_i \geq \mu,\ \sum_i \lambda_i = 1\}$ by Proposition 5. This set is a simplex whose vertices have eight entries μ and one entry $1 - 8\mu$. The

concave function $\sum_i \lambda_i^p$ attains its minimum at a vertex, which gives $s_2 \geq B(p)$. At $p = 1/1000$ the inequality $\frac{3}{2}B(p) > \log 26$ is equivalent to $X^{500} > 26^{333}$ with $X = 8\mu^{1/1000} + (1-8\mu)^{1/1000}$. The program certifies it with rational lower bounds on the two roots and exact big-integer comparisons; numerically $\frac{3}{2}B(1/1000) = 4.7367\ldots > \log 26 = 4.7004\ldots$. As $p \to 0^+$, $B(p) \to \log 9$ and $\frac{3}{2}\log 9 = \log 27 > \log 26$; continuity gives the last claim. Finally, at $p = 1/1000$ the state $\tau_\star = \mathrm{diag}(1 - 2^{-20}, 2^{-21}, 2^{-21})$ satisfies $\frac{1}{3}\log 26 < S_p(\tau_\star) \leq \frac{1}{2}B(p)$: with $T = \mathrm{Tr}\,\tau_\star^p$ these are $T^{1000} > 26^{333}$ and $T^2 \leq X$, which the program certifies in the same way. Numerically $S_p(\tau_\star) = 1.5726\ldots$ lies in the window $(1.5668\ldots, 1.5789\ldots]$. □

Proof of Theorem 1. (a) By Lemma 8, at $p = 1/1000$ we have $s_3/3 \leq \frac{1}{3}\log 26 < S_p(\tau_\star) \leq \frac{1}{2}B(p) \leq s_2/2$, so Lemma 3 applies with $b = S_p(\tau_\star)$ and gives (1) with $m = 3$. (b) By Lemma 8, $s_3 \leq \log 26 < \frac{3}{2}B(p) \leq \frac{3}{2}s_2$ for every sufficiently small p, so the window $(s_3/3, s_2/2]$ is nonempty, a state τ_p with $S_p(\tau_p)$ in it exists, and Lemma 3 applies. In both cases Φ_τ is completely positive and trace preserving: its Kraus operators are $K_j\Pi \otimes |0\rangle$ and $\sqrt{t_i}\,|t_i\rangle\langle e| \otimes |1\rangle$ for a spectral decomposition $\tau = \sum_i t_i |t_i\rangle\langle t_i|$. □

7 Verification

The file `check_delayed_onset.py` is standalone. It uses only the Python standard library, exact arithmetic in $\mathbb{Z}[\omega]$ and big integers, and no network, and it takes no input. Its output is fixed in `expected_stdout.txt`. It checks the design and trace preservation; the identity $h = x^\dagger W x$ on exact test vectors; and $P \succ 0$ by two independent exact routes. The first route rounds P to a Hermitian matrix, bounds the rounding error in Frobenius norm, and shows that all 144 leading principal minors of the shifted rounded matrix are positive integers (fraction-free elimination over $\mathbb{Z}[\omega]$). The second route rebuilds P with independent partial-transpose code, exhibits an embedded Cholesky-type factor G' with entries in $2^{-24}\mathbb{Z}[\omega]$, and proves $\|P - G'G'^\dagger - c_0 I\|_F < 2^{-19} < c_0$ by an exact integer comparison. The program also checks the negative control with all $\theta_j = 1$, all 216 identities of Proposition 7, and the inequality $X^{500} > 26^{333}$ of Lemma 8. The program prints 21 lines in under a second, and its output is byte-identical across isolated runs. Thirty single-point mutations each make it exit with a nonzero status. They include corruptions of both certificates, a larger c_0, the wrong partial-transpose factor, a missing conjugate, $\theta_6 = 1$, a sign flip in φ, weakened bounds at $p = 1/1000$ and constant blocks outside the admissible window. The harness `mutate.py` and its report are included. The certificate R was found by a numerical semidefinite program and is embedded verbatim; the Gram factor G' was produced by `build_cert.py` (numerical, at build time only). The program trusts neither and verifies everything exactly. Reproduction establishes only that the program computes what it prints. Lemmas 2 and 3, the analytic part of Lemma 8, and the reduction from $x^\dagger W x$ to output ranks are established by the written argument; the numerical inequalities of Lemma 8 are certified by the program.

8 Discussion

The first violation here is a rank effect, which is why it needs small p. At $p \to 0$, $S_{p,\min}$ becomes the logarithm of the minimum output rank. The channel Ψ has minimum output rank 3 on one copy and $9 = 3^2$ on two, so it is multiplicative up to two copies, while it is at most $26 < 27$ on three. Cubitt, Harrow, Leung, Montanaro and Winter exhibited an explicit pair of different channels from 4 to 3 dimensions with non-multiplicative minimum output rank, and proved small-p violations by a random construction [3]. The new feature here is that the rank of a single channel stays multiplicative at two copies and first fails at three. The flag does real work: for Ψ itself we only know $s_2 = 2s_1$ in the limit $p \to 0$, and the constant channel τ absorbs the difference at positive p, so that exact two-copy additivity holds for Φ.

Three questions remain open. The first is whether a delayed first violation exists for $p \geq 1$, in particular for the von Neumann entropy. The second is whether every order $m \geq 3$ can be the first violating one. The third is whether a single channel without a flagged constant block can exhibit the phenomenon at fixed $p > 0$.

AI assistance and review status

AI tools were used to develop the argument, to write the verification program, to assist with exposition and to conduct internal adversarial reviews. The manuscript follows the evidence-tracking guidance of Scientific Agent Skills [8]. These activities are disclosed as AI assistance, not as external refereeing or formal proof verification.

An adversarial review was performed by AI in an isolated context. It rebuilt the matrix W from the Kraus operators and recomputed P exactly over $\mathbb{Z}[\omega]$. It proved $P \succ 0$ by its own exact Gram certificate, independently of the elimination route. It tested the partial-transpose convention against alternatives that would silently fail. It re-derived the three-copy identities, the entropy bound at $p = 1/1000$ and the flagged-sum formula. The reductions that the argument uses were developed over several internal research rounds, each checked internally. A final audit of this manuscript and its program recomputed every number in it independently and led to the explicit choice of $\tau_\star$ in Theorem 1(a). **No independent human expert has examined this argument, and no novelty certification by a human is recorded.** Searches of arXiv through 27 September 2026 found no prior delayed first violation. The two most recent related preprints [5, 6] concern two-use violations and gaps. The problem record [7] was still listed as unsolved when this manuscript was prepared.

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