Anticoncentration of Independent Complex Gaussian Hafnians

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Abstract

Let X be a complex symmetric $2n \times 2n$ matrix with zero diagonal and independent standard circular complex Gaussian entries above the diagonal. We prove a uniform shifted small-ball bound $\mathbb{P}(|\operatorname{Haf}(X) - z| \le \varepsilon\sqrt{(2n-1)!!}) \le \min\{1, (2/\sqrt{\pi})\sqrt{n}\,\varepsilon^2\}$. The proof bounds the inverse conditional variance by $1/(2n-2)!!$. A complex Gaussian bilinear identity and an exact partition of perfect matchings permit coordinate compression, giving a recursive comparison of Laplace transforms. This adapts the Gaussian-kernel/compression method of Koehler–Leung and Zhao to the independent complex ensemble. In particular, the polynomial $p(n,u) = 2nu$ satisfies the all-dimension lower-tail question in QIQCOP problem `op_55be40726cdf7304`. The result concerns independent symmetric entries, not Gaussian Gram matrices, and does not establish an average-case hardness theorem.

1 Statement and relation to earlier work

A standard circular complex Gaussian $Z \sim \mathcal{CN}(0,1)$ has planar density $\pi^{-1}e^{-|z|^2}$, so $\mathbb{E}|Z|^2 = 1$. For a symmetric matrix of even size, its hafnian is the sum over perfect matchings of the product of the matched entries; set $\operatorname{Haf}(\varnothing) = 1$. Let X_n have independent $\mathcal{CN}(0,1)$ entries above its zero diagonal, with size $2n$. Put

$$H_n = \operatorname{Haf}(X_n), \qquad h_n = (2n-1)!!, \qquad V_n = \sum_{j=2}^{2n} |\operatorname{Haf}(X_n[\{1,j\}^c])|^2. \tag{1.1}$$

The complement here is within $\{1, \ldots, 2n\}$. Distinct matching monomials are orthogonal: in the product of one monomial and the conjugate of a different one, some independent centered entry occurs on only one side. Thus $\mathbb{E}|H_n|^2 = h_n$.

Hafnian anticoncentration is motivated by Gaussian boson sampling [2, discussion following Eq. (11)]. Koehler and Leung develop a Gaussian-kernel and coordinate-compression method for permanents over real, complex, and quaternionic Ginibre ensembles [3]. Zhao proves a shifted bound for real Gram hafnians and, by a limit, independent real symmetric hafnians [4, Theorem 2.3]. The complex Gram ensemble in Ref. [5] has a different dependence structure. We adapt the kernel and matching arguments to the independent circular complex model, proving the needed identities directly rather than transferring a real-ensemble theorem. The precise catalog formulation is in Ref. [1].

Theorem 1. *For every integer $n \geq 1$,*

$$\mathbb{E}[V_n^{-1}] \leq \frac{1}{(2n-2)!!}, \qquad 0!! = 1. \tag{1.2}$$

For every $z \in \mathbb{C}$ and $\varepsilon \geq 0$,

$$\begin{aligned}\mathbb{P}(|H_n - z| \leq \varepsilon\sqrt{h_n}) &\leq \min\left\{1, \frac{h_n}{(2n-2)!!}\varepsilon^2\right\} \\ &\leq \min\left\{1, \frac{2}{\sqrt{\pi}}\sqrt{n}\,\varepsilon^2\right\}.\end{aligned} \tag{1.3}$$

Consequently the polynomial $p(n,u) = 2nu$ satisfies

$$\mathbb{P}\left(|H_n| < \frac{\sqrt{h_n}}{p(n,1/\delta)}\right) < \delta \quad (n \geq 1,\ 0 < \delta < 1). \tag{1.4}$$

2 A Gaussian bilinear identity

Lemma 2. *For independent $x, y \sim N(0, I_d)$, a real $d \times d$ matrix T, and real vectors a, b, put $D = I + TT^\mathsf{T}$ and $E = I + T^\mathsf{T}T$. Then*

$$\begin{aligned}F_T(a,b) &:= \mathbb{E}e^{i(x^\mathsf{T}Ty + x^\mathsf{T}a + y^\mathsf{T}b)} \\ &= \det(E)^{-1/2}\exp\Big(-\tfrac{1}{2}a^\mathsf{T}D^{-1}a - \tfrac{1}{2}b^\mathsf{T}E^{-1}b - ia^\mathsf{T}TE^{-1}b\Big).\end{aligned} \tag{2.1}$$

Both $F_T(a,0)$ and $F_T(0,b)$ are positive, and for $0 \leq \theta \leq 1$,

$$|F_T(\sqrt{\theta}a, \sqrt{1-\theta}b)| = F_T(a,0)^\theta F_T(0,b)^{1-\theta}. \tag{2.2}$$

Proof. Integration over x gives $\exp(-|Ty+a|^2/2 + iy^\mathsf{T}b)$. Completing the Gaussian square in y and using $I - TE^{-1}T^\mathsf{T} = D^{-1}$ proves (2.1). Its mixed term is purely imaginary, giving (2.2). ☐

For independent $X, Y \sim \mathcal{CN}(0, I_d)$, let $x = \sqrt{2}(\Re X, \Im X)$ and $y = \sqrt{2}(\Re Y, \Im Y)$. For a complex matrix K and real t,

$$t\Re(X^\mathsf{T}KY) = x^\mathsf{T}Ty, \qquad T = \frac{t}{2}\begin{pmatrix}\Re K & -\Im K \\ -\Im K & -\Re K\end{pmatrix}. \tag{2.3}$$

A linear term $t\Re(X^\mathsf{T}h)$ corresponds to $a = (t/\sqrt{2})(\Re h, -\Im h)$. These formulas place the complex bilinear expressions below within Lemma 2.

3 Fixed-row phase reduction and matching expansion

Fix $n \geq 2$, let J be the $2n-1$ vertices other than vertex 1, and fix its row $y \in \mathbb{C}^J$. All internal edges on J remain independent $\mathcal{CN}(0,1)$. Denote the resulting hafnian by H_y and write $\Phi_y(t) = \mathbb{E}e^{it\Re H_y}$. Every matching contains exactly $n-1$ internal edges. Multiplying all of those random edges by $e^{i\alpha/(n-1)}$ preserves their joint law and multiplies the hafnian by $e^{i\alpha}$. Therefore H_y is circular.

Write $y_j = |y_j|e^{i\phi_j}$, choosing any phase at a zero coordinate. The substitution $W_{jk} \mapsto e^{i(\phi_j+\phi_k)}W_{jk}$ preserves the internal-edge distribution. A matching paired with vertex j in the first row uses all internal vertices except j, and hence

$$H_y(W \text{ after substitution}) = e^{i\sum_{j\in J}\phi_j}H_{|y|}(W). \tag{3.1}$$

Circularity removes the global phase, so $\Phi_y = \Phi_{|y|}$. It suffices to work with nonnegative real row coefficients.

Choose distinct $p, q \in J$, put $R = J \setminus \{p, q\}$, and condition on all internal edges of R, denoted by W. For $a \in R$, define

$$h_a = \mathrm{Haf}(W[R \setminus \{a\}]), \qquad C = \sum_{i \in R} y_i h_i, \tag{3.2}$$

$$K_{aa} = 0, \qquad K_{ab} = \sum_{i \in R \setminus \{a,b\}} y_i \,\mathrm{Haf}(W[R \setminus \{i, a, b\}]) \quad (a \neq b). \tag{3.3}$$

Empty sums are zero. Write $X_a = W_{pa}$, $Y_a = W_{qa}$, and $Z = W_{pq}$ for the still unexposed edges. Conditional on W, the vectors X, Y and the scalar Z are independent standard circular Gaussians, and

$$H_y = X^\mathsf{T} K Y + y_q X^\mathsf{T} h + y_p Y^\mathsf{T} h + ZC. \tag{3.4}$$

Indeed, pairing vertex 1 with q or p gives the two linear terms. Pairing it with $i \in R$ either pairs p with q, contributing to ZC, or pairs p, q with distinct vertices $a, b \in R \setminus \{i\}$, contributing to $X^\mathsf{T} K Y$. These cases are disjoint and exhaustive. For $n = 2$, $|R| = 1$ and $K = 0$, so the identity remains valid. The correlated objects K, h, C are fixed by the conditioning and do not depend on y_p, y_q.

4 Coordinate compression

Suppose $y_p, y_q > 0$, set $r = \sqrt{y_p^2 + y_q^2}$ and $\theta = y_q^2/r^2$, and form two new rows: $y^{(q)}$ replaces (y_p, y_q) by $(0, r)$, while $y^{(p)}$ replaces it by $(r, 0)$. All other coordinates remain fixed. Conditional on W, integration over Z in (3.4) contributes the same positive factor $e^{-t^2|C|^2/4}$ for all three rows. Apply (2.3) and Lemma 2 to X, Y. Let J_q, J_p be the conditional characteristic functions for the two endpoints. They are positive, and (2.2) gives

$$\left|\mathbb{E}(e^{it\Re H_y} \mid W)\right| = J_q^\theta J_p^{1-\theta}. \tag{4.1}$$

The common Z factor is preserved since the exponents sum to one. Triangle and Hölder inequalities now give

$$|\Phi_y(t)| \leq \Phi_{y^{(q)}}(t)^\theta \Phi_{y^{(p)}}(t)^{1-\theta} \leq \max\{\Phi_{y^{(q)}}(t), \Phi_{y^{(p)}}(t)\}. \tag{4.2}$$

Both endpoint values are nonnegative, being averages of positive quantities.

Put $\phi_m(t) = \mathbb{E}e^{it\Re H_m}$. Conditional on its internal edges, first-row expansion makes H_m a circular Gaussian of variance V_m. Consequently, even before showing that this variance is positive,

$$\phi_m(t) = \mathbb{E}e^{-t^2 V_m/4} \geq 0. \tag{4.3}$$

At a fixed t, repeatedly choose the larger endpoint in (4.2). Each step preserves the row norm and decreases its support size by one. A singleton row of norm r leaves precisely the independent ensemble of size $2n-2$, so its hafnian has law rH_{n-1}. Together with phase reduction, this proves

$$|\Phi_y(t)| \leq \phi_{n-1}(\|y\|t) \qquad (n \geq 2). \tag{4.4}$$

The zero row is immediate. Endpoint choices may depend on t; the common singleton value makes the resulting pointwise bound sufficient. Distinct hafnian minors have nowhere been assumed independent.

5 Inverse variance and shifted disks

For a random first row, $G = \|y\|^2 \sim \mathrm{Gamma}(2n-1, 1)$, independently of the internal edges. Averaging (4.4) and using (4.3) shows that for every $s \geq 0$,

$$\mathbb{E}e^{-sV_n} \leq \mathbb{E}e^{-sGV_{n-1}}, \tag{5.1}$$

where the two variables on the right are independent copies. Integrating over $s \geq 0$ and using Tonelli's theorem, with $1/0 = +\infty$, yields

$$\mathbb{E}V_n^{-1} \leq \mathbb{E}G^{-1}\mathbb{E}V_{n-1}^{-1} = \frac{1}{2n-2}\mathbb{E}V_{n-1}^{-1}. \tag{5.2}$$

The base case is $V_1 = |\operatorname{Haf}(\varnothing)|^2 = 1$. Induction proves (1.2) and in particular $V_n > 0$ almost surely.

Conditional on internal edges, the density of H_n is bounded above by $1/(\pi V_n)$. Every disk of radius a has conditional probability at most a^2/V_n, irrespective of its center. Set $a = \varepsilon\sqrt{h_n}$ and average to obtain the first inequality of (1.3). Finally,

$$\frac{h_n}{(2n-2)!!} = \frac{2\Gamma(n+1/2)}{\sqrt{\pi}\Gamma(n)} \leq \frac{2}{\sqrt{\pi}}\sqrt{n}. \tag{5.3}$$

For the inequality, apply Jensen to $\mathbb{E}\sqrt{Q}$ with $Q \sim \mathrm{Gamma}(n, 1)$ and $\mathbb{E}Q = n$. Taking $\varepsilon = \delta/(2n)$ bounds the probability of the closed disk containing the event in (1.4) by $\delta^2/(2\sqrt{\pi}n^{3/2}) < \delta$. This proves Theorem 1, including $n = 1$ and all $0 < \delta < 1$.

The theorem is analytic. Supplementary exact finite matching identities check algebraic transcription only; neither finite experiments nor moment sampling establish the all-n inverse-moment comparison. The conclusion addresses the catalog's independent complex ensemble. It does not transfer a bound to correlated Gram entries or prove the separate complexity assumptions used in sampling arguments.

Verification, provenance, and scope

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