CGLMP measurements need not maximize statistical strength

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Version 1.0 — 28 September 2026

Abstract

We give a four-dimensional counterexample to the conjecture that the standard CGLMP measurements maximize relative-entropy statistical strength on a fixed maximally entangled state. Identify each local four-dimensional space with two qubits and use two CHSH experiments with a shared setting label. This gives two rank-one, four-outcome projective measurements per party on the prescribed state $|\Phi_4\rangle$. Their statistical strength is greater than 0.087 bits, whereas the CGLMP measurements have strength less than 0.065 bits even after optimizing the setting distribution. The lower bound applies to the entire four-outcome local polytope; it does not assume independent local responses on the component qubits. The upper bound follows from an explicit local mixture valid in every setting pair. A short integer-and-rational certificate proves the strict separation. This refutes the universal statistical-strength equality in QIQCOP problem `op_a34f0e2d6489068f`.

1 Question and result

A Bell experiment can be compared with local realism by the relative entropy between its outcome distribution and the best local model. This statistical strength measures an asymptotic evidence rate, rather than the value of a particular linear Bell functional [1]. Acín, Gill and Gisin studied its optimization numerically and found CGLMP measurement choices in low dimensions [2]. Gill subsequently described the optimality suggested by searches with a fixed maximally entangled state as conjectural [3, Section 4]. We address the precise fixed-state equality recorded in the QIQCOP catalog [4].

There are two settings $x, y \in \{0, 1\}$ per party and d outcomes per measurement. Write $\mathcal{L}$ for the convex hull of all deterministic local behaviors in this scenario. For a conditional behavior p and a distribution μ on the four setting pairs, define

$$S(p; \mu) = \inf_{\ell \in \mathcal{L}} \sum_{x,y,a,b} \mu(x, y) p(a, b \mid x, y) \log_2 \frac{p(a, b \mid x, y)}{\ell(a, b \mid x, y)}, \tag{1}$$

$$S^\star(p) = \sup_{\mu \in \Delta(\{0,1\}^2)} S(p; \mu). \tag{2}$$

The usual relative-entropy conventions apply at zero probabilities. The shared state is fixed to

$$|\Phi_d\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j,j\rangle. \tag{3}$$

The catalog specifies the CGLMP bases as

$$\begin{aligned} |a;x\rangle &= \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{2\pi i j(a+\alpha_x)/d} |j\rangle, \\ |b;y\rangle &= \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{2\pi i j(-b+\beta_y)/d} |j\rangle, \end{aligned} \tag{4}$$

with $\alpha_0 = 0$, $\alpha_1 = -1/2$, $\beta_0 = 1/4$ and $\beta_1 = 3/4$. Let p_C be their behavior. The question is whether, for every $d \geq 3$,

$$S^\star(p_C) = \sup_M S^\star(p_M), \tag{5}$$

where M ranges over two projective d-outcome measurements per party on $|\Phi_d\rangle$.

Theorem 1. *For $d = 4$, there are two rank-one four-outcome projective measurements per party on $|\Phi_4\rangle$ whose behavior p_T satisfies*

$$S^\star(p_T) > \frac{87}{1000}, \qquad S^\star(p_C) < \frac{65}{1000}. \tag{6}$$

Consequently $S^\star(p_T) - S^\star(p_C) > 22/1000$ bits, and (5) is false.

The proof needs neither the exact strength of either behavior nor a global optimization over measurements. A lower bound for one admissible measurement and an upper bound for the specified CGLMP measurements are enough.

2 Two CHSH components with the same settings

Identify $\mathbb{C}^4$ locally with $\mathbb{C}^2 \otimes \mathbb{C}^2$ by the binary computational basis. With the bipartite tensor factors regrouped, $|\Phi_4\rangle$ is $|\Phi_2\rangle \otimes |\Phi_2\rangle$. On each component qubit Alice uses

$$A_0 = Z, \qquad A_1 = X, \tag{7}$$

and Bob uses

$$B_0 = \frac{Z+X}{\sqrt{2}}, \qquad B_1 = \frac{Z-X}{\sqrt{2}}. \tag{8}$$

Here X, Z are the Pauli matrices. For setting x, Alice measures both her qubits in the eigenbasis of A_x; Bob does the same with B_y. Each party reports the two eigenvalue bits as one four-valued outcome. Thus the setting alphabet remains $\{0,1\}$ at each party. The four product basis vectors at each setting define rank-one orthogonal projectors summing to the identity.

Let bit 0 denote eigenvalue $+1$. A component wins when $a_i \oplus b_i = xy$, for $i = 1,2$, and let K be the number of wins. Choose the four setting pairs uniformly. In each setting pair a component wins with probability $w = (2+\sqrt{2})/4$. Conditional independence of the two quantum components gives

$$\Pr(K=0) = P_0 = \frac{3-2\sqrt{2}}{8}, \quad \Pr(K=1) = \frac{1}{4}, \quad \Pr(K=2) = P_2 = \frac{3+2\sqrt{2}}{8}. \tag{9}$$

These probabilities do not depend on the setting pair.

For a deterministic local four-outcome strategy, each component bit defines a deterministic binary CHSH strategy. Such a strategy wins at most three of the four setting pairs: the four proposed winning equations cannot all hold, since their left sides have even parity and their right sides have odd parity. It follows, component by component and then by convexity, that

$$\mathbb{E}_\ell K \leq \frac{3}{2} \qquad (\ell \in \mathcal{L}). \tag{10}$$

In this argument the two response bits may have arbitrary correlations. Equation (10) therefore constrains the full four-outcome local polytope, not just a product subset.

Lemma 1. *For probability distributions P, Q on a finite set and any strictly positive function f,*

$$D(P\|Q) \geq \mathbb{E}_P \log_2 f - \log_2 \mathbb{E}_Q f. \tag{11}$$

Proof. If Q vanishes on the support of P, the left side is infinite. Otherwise set $\widetilde{Q} = Qf/\mathbb{E}_Q f$. Expanding $D(P\|\widetilde{Q}) \geq 0$ gives the inequality. □

Apply the lemma to the joint setting-and-outcome distributions $P(x, y, a, b) = p_T(a, b \mid x, y)/4$ and $Q(x, y, a, b) = \ell(a, b \mid x, y)/4$, with

$$f(K) = 1 + \frac{K - 3/2}{2}, \qquad (f(0), f(1), f(2)) = \left(\frac{1}{4}, \frac{3}{4}, \frac{5}{4}\right). \tag{12}$$

Equation (10) gives $\mathbb{E}_Q f \leq 1$ for every local behavior. Thus

$$S^\star(p_T) \geq S(p_T; \text{uniform}) \geq L, \qquad L = -2P_0 + \frac{1}{4}\log_2 \frac{3}{4} + P_2 \log_2 \frac{5}{4}. \tag{13}$$

Section 4 certifies $L > 87/1000$.

3 An explicit local model above the CGLMP strength

All output arithmetic in this section is modulo four. For the phases in (4), define

$$t_{00} = a - b, \quad t_{01} = -a + b - 1, \quad t_{10} = -a + b, \quad t_{11} = a - b. \tag{14}$$

The amplitude on $|\Phi_4\rangle$ is the geometric sum $\frac{1}{8}\sum_{j=0}^{3} e^{-2\pi i j(a-b+\alpha_x+\beta_y)/4}$. Its squared modulus is

$$p_C(a, b \mid x, y) = \frac{q(t_{xy})}{4}, \qquad q(t) = \frac{1}{32\sin^2(\pi(t + 1/4)/4)}. \tag{15}$$

Indeed, the four offsets in the amplitude are respectively $t_{00} + 1/4$, $-(t_{01} + 1/4)$, $-(t_{10} + 1/4)$ and $t_{11} + 1/4$ modulo four. This also checks the phase convention directly.

We construct one local behavior of the form

$$\ell(a, b \mid x, y) = \frac{r(t_{xy})}{4}, \qquad r = \left(\frac{7}{10}, \frac{1}{20}, \frac{1}{20}, \frac{1}{5}\right). \tag{16}$$

With probability $4/5$, choose a uniformly random permutation of $(0, 0, 0, 3)$; with probability $1/5$, choose a uniformly random permutation of $(0, 0, 1, 2)$. Call the resulting four entries $(t_{00}, t_{01}, t_{10}, t_{11})$. Their sum is -1 modulo four. Set

$$a_0 = 0, \qquad b_0 = -t_{00}, \qquad b_1 = t_{01} + 1, \qquad a_1 = t_{11} + b_1. \tag{17}$$

The first, second and fourth differences in (14) then agree by construction. The third agrees because $-a_1 + b_0 = -t_{11} - t_{01} - 1 - t_{00} = t_{10}$ modulo four. Finally add an independent uniform shift in $\{0, 1, 2, 3\}$ to all four outputs. This defines a mixture of deterministic local response functions.

Every coordinate of the randomly permuted tuple has marginal

$$\frac{4}{5}\left(\frac{3}{4}, 0, 0, \frac{1}{4}\right) + \frac{1}{5}\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0\right) = r. \tag{18}$$

The common shift makes the four outcome pairs with a specified difference equiprobable. This proves (16) in every setting pair simultaneously. The divergence to this same local behavior in each pair is

$$U = \sum_{t=0}^{3} q(t) \log_2 \frac{q(t)}{r(t)}. \tag{19}$$

Consequently $S(p_C; \mu) \le U$ for every μ, including distributions on the boundary of the setting simplex. Taking the supremum gives $S^\star(p_C) \le U$. The certified inequality $U < 65/1000$, together with (13), proves Theorem 1.

4 Exact arithmetic and reproducibility

The strict inequalities can be certified using only integers and rational numbers. For an algebraic expression for (15), put $u = \sqrt{2 + \sqrt{2}}$ and $v = \sqrt{2 - \sqrt{2}}$. Then

$$(q(0), q(1), q(2), q(3)) = \left(\frac{1}{8(2-u)}, \frac{1}{8(2+v)}, \frac{1}{8(2+u)}, \frac{1}{8(2-v)}\right). \tag{20}$$

The accompanying `exact-check.py` encloses each square root by integer square-root arithmetic with denominator 10^{18}. For $x \ge 1$, let $z = (x-1)/(x+1)$. It uses

$$\ln x = 2 \sum_{j=0}^{n-1} \frac{z^{2j+1}}{2j+1} + R_n, \qquad 0 \le R_n \le \frac{2z^{2n+1}}{(2n+1)(1-z^2)}, \tag{21}$$

with $n = 80$, and uses reciprocal symmetry for $0 < x < 1$. Monotonicity and outward interval operations then bound (13) and (19). The checker asserts and prints all three strict comparisons in Theorem 1. It also checks the local tuple construction and the CHSH win bound for every one of the $4^4 = 256$ deterministic local four-outcome strategies.

For orientation, the quantities being bounded are

$$L \approx 0.0878892114243, \qquad U \approx 0.0627382117716. \tag{22}$$

These are the displayed lower and upper bounds, not assertions of the exact optimized statistical strengths. The theorem uses only the coarser strict rational comparisons. A separate 90-digit matrix calculation checks the rank-one bases and all 64 conditional probability entries against the formulas above. An exact rational calculation checks all 64 entries of the local mixture. Those independent controls supplement the analytic proof; floating-point output is not its certificate.

5 Relation to prior work and status

The setting optimization in (1) allows correlated settings, as in Definition 3 of [1]. The same separation also holds if one restricts to independent setting distributions: the candidate uses uniform settings, and the CGLMP upper bound holds for every distribution.

The numerical optimization in [2] treats both fixed-state and joint state–measurement questions. Its best reported four-dimensional state is not maximally entangled. The discussion on page 4 also considers two independent copies of a four-dimensional experiment, leading to four settings and sixteen outcomes per party. That observation is relevant background. Here there are only two setting labels per party, each used on both component qubits, and the local dimension is four. The proof bounds all local four-outcome responses directly and does not invoke additivity of statistical strength. Gill's later discussion [3, Section 4] provides the conjectural numerical motivation for the catalog question.

Theorem 1 supplies a negative answer to that exact universal equality. It does not classify optimal measurements in other dimensions, settle the maximal linear CGLMP violation, or classify Bell-polytope facets. A focused comparison with the three primary sources above found no matching certified fixed-state comparison, but does not establish historical priority.

Provenance and verification. This result arose in round 48 of an AI-assisted QIQC research campaign. GPT-6 Astra, through the Inspect AI rolling-state workflow, generated the candidate. Two subsequent frozen AI reviews found no defect in the original scope or proof. Integer-and-rational reproduction and an independently written physical-probability check passed. These are internal verification records, not independent human peer review or catalog acceptance. The scientific-writing skill from Scientific Agent Skills [5] guided evidence tracking and exposition. The proof, code, source comparison and review records accompany this version.

References

[1] W. van Dam, R. D. Gill and P. D. Grünwald, The statistical strength of nonlocality proofs, *IEEE Transactions on Information Theory* **51**, 2812–2835 (2005). doi:10.1109/TIT.2005.851738. arXiv:quant-ph/0307125.

[2] A. Acín, R. Gill and N. Gisin, Optimal Bell tests do not require maximally entangled states, *Physical Review Letters* **95**, 210402 (2005). doi:10.1103/PhysRevLett.95.210402. arXiv:quant-ph/0506225.

[3] R. D. Gill, Better Bell inequalities (passion at a distance), *IMS Lecture Notes–Monograph Series* **55**, 135–148 (2007). doi:10.1214/074921707000000328. arXiv:math/0610115.

[4] QIQCOP Zoo, Statistical strength of CGLMP measurements, problem `op_a34f0e2d6489068f`, statement updated 9 September 2026, accessed 28 September 2026. `https://qiqc-op.com/problem/op_a34f0e2d6489068f/`.

[5] T. Kassis, V. Agarwal, Y. He, D. Patel and A. M. Brueckner, Scientific Agent Skills: A Library of Procedural Knowledge for Research Agents (2026). arXiv:2609.00065.