Operator Schmidt rank under bounded spectral entangling strength

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17 November 2026 Version 1.0

Abstract

We consider a finite bipartite Hamiltonian whose interaction has bounded ancilla-assisted spectral entangling strength, with no restriction on either local Hamiltonian. At fixed time and fixed operator-norm error, its full time-evolution operator admits an approximation of operator Schmidt rank $O((\log D)^2)$, where D is the total Hilbert-space dimension. This rules out the positive-power rank growth asked about after Proposition 9 of Kim et al., Phys. Rev. X **16**, 041035 (2026). The central step is a dimension-independent approximate factorization theorem: a Hermitian operator of bounded spectral entangling strength can be approximated by a tensor decomposition with all four local row and column square sums bounded. A completely positive dilation supplies local positive weights. Fourier bands of their dyadic spectral decompositions obey two complementary decay bounds, whose combination gives a uniform truncation error. Applying this factorization to the two Hermitian parts of the full unitary and averaging Gaussian product operators gives the rank bound. The proof also supplies an explicit, nonoptimal dependence on time and accuracy.

1 The dimension dependence of operator approximation

How much of a quantum evolution can be recovered from its ability to create entanglement? A bound on the Schmidt spectrum of evolved states permits state-dependent truncation, but approximating the entire evolution in operator norm is a different requirement. Kim et al. [1] introduced spectral entangling strength to quantify this distinction. Their Proposition 9 gives bounded-strength dynamics for which no fixed operator Schmidt rank works uniformly in dimension. They ask whether the required rank can grow as a positive power of the Hilbert-space dimension at fixed time and accuracy. We prove a squared-logarithmic upper bound.

Let $\mathcal{H}_A, \mathcal{H}_B$ be finite-dimensional complex Hilbert spaces with dimensions d_A, d_B, and write $D = d_A d_B$. For a bipartite vector, $\|\psi\|_{\mathrm{Sch},1}$ denotes the sum of its Schmidt coefficients. For an operator X on $\mathcal{H}_A \otimes \mathcal{H}_B$, define

$$\mathcal{J}(X) = \sup_{\alpha,\beta} \|(X \otimes I_{A'B'})|\alpha\rangle_{AA'}|\beta\rangle_{BB'}\|_{\mathrm{Sch},1}\,, \tag{1}$$

where the supremum is over normalized local vectors and arbitrary finite local ancillas. This is the strength $\bar{\mathcal{J}}$ of Definition 1 in Ref. [1]. The operator Schmidt rank $\mathrm{OSR}(X)$ is the least number of product operators in a decomposition of X. All unqualified operator norms below are spectral norms, and log denotes the natural logarithm.

Theorem 1. *Let*

$$H = H_A \otimes I_B + I_A \otimes H_B + V, \qquad V = V^*, \qquad \mathcal{J}(V) \le 1, \tag{2}$$

with arbitrary finite-dimensional Hermitian H_A, H_B. For every $t \in \mathbb{R}$ and $0 < \epsilon \leq 1$, there is an operator $\widetilde{U}$ such that

$$\left\| e^{-itH} - \widetilde{U} \right\| \leq \epsilon, \qquad \mathrm{OSR}(\widetilde{U}) \leq C_{t,\epsilon} [\log(2D)]^2. \tag{3}$$

The constant is independent of d_A, d_B, H_A, H_B. More explicitly, with $T = |t|$ and $q = 1 + T + \log(1/\epsilon)$, a universal constant C gives

$$\mathrm{OSR}(\widetilde{U}) \leq C e^{10T} \epsilon^{-10} q^{10} \log(2D)[\log(2D) + q]. \tag{4}$$

The approximation is for the full operator on AB, and $\widetilde{U}$ may be nonunitary. The theorem proves existence of a short tensor representation; it does not bound the cost of finding its factors. A fixed strength bound $\mathcal{J}(V) \leq J_0$ is handled by rescaling time to $J_0 t$. When $J_0 = 0$, the evolution is local and has rank one.

For comparison, Mackay et al. [5] obtain sharper compression results under spatial locality and bounded local-strength assumptions. Their setup has bounded on-site dimension and finite-range or exponentially decaying interactions. No such structure is assumed in Theorem 1. The Note added in Ref. [1] also announces an operator approximation theorem for Hermitian operators with bounded α-SE strength for $\alpha < 1/2$. The norm in (1) is the endpoint $\alpha = 1/2$; that announcement does not state the endpoint result proved here.

The proof separates two tasks. First we approximate a Hermitian interaction by a decomposition whose local factors have bounded row and column square sums. Exact factorizations controlled at the two endpoints need not be the same factorization. We reconcile them after a truncation chosen from local positive weights. Second, we use the known bound on the strength of the entire evolution and apply the factorization to its Hermitian parts. One Gaussian product sample has rank one, and matrix concentration turns its dimension-independent variance into a squared-logarithmic rank bound.

2 Norms and a completely positive dilation

We begin with the norm interfaces, keeping track of the two adjoints used later. If $\psi = \sum_r s_r \alpha_r \otimes \beta_r$ is a Schmidt decomposition, the triangle inequality in (1) gives

$$\|(X \otimes I)\psi\|_{\mathrm{Sch},1} \leq \mathcal{J}(X) \sum_r s_r. \tag{5}$$

Consequently $\mathcal{J}(XY) \leq \mathcal{J}(X)\mathcal{J}(Y)$ and $\mathcal{J}(I) = 1$. Multiplication on either side by a product unitary preserves $\mathcal{J}$.

Lemma 2. *For any bipartite operator X,*

$$\mathcal{J}(X) = \inf_{X = \sum_s A_s \otimes B_s} \left\| \sum_s A_s^* A_s \right\|^{1/2} \left\| \sum_s B_s^* B_s \right\|^{1/2}. \tag{6}$$

Moreover,

$$\|X\| \leq \sqrt{\mathcal{J}(X)\mathcal{J}(X^*)}. \tag{7}$$

Proof. Realign X into the matrix

$$R(X)_{(a,c),(b,d)} = X_{(a,b),(c,d)}. \tag{8}$$

Local Schmidt decompositions of the input vectors in (1) reduce the ancillary optimization to density matrices ρ, σ on the physical input spaces. Local ancillary isometries preserve the output Schmidt coefficients. With harmless transposes absorbed into the density variables, this gives

$$\mathcal{J}(X) = \max_{\rho,\sigma} \left\| (I_A \otimes \sqrt{\rho}) R(X) (I_B \otimes \sqrt{\sigma}) \right\|_1 . \tag{9}$$

Consider the finite semidefinite program

$$\begin{aligned} &\text{minimize} && (x+y)/2, \\ &\text{subject to} && \begin{pmatrix} P & R(X) \\ R(X)^* & Q \end{pmatrix} \succeq 0, \\ & && \operatorname{Tr}_{A,\text{out}} P \preceq x I_A, \qquad \operatorname{Tr}_{B,\text{out}} Q \preceq y I_B. \end{aligned} \tag{10}$$

Its dual maximizes $\operatorname{Re}\operatorname{Tr}(R(X)^* K)$ over density matrices ρ, σ and K satisfying

$$\begin{pmatrix} I_A \otimes \rho & K \\ K^* & I_B \otimes \sigma \end{pmatrix} \succeq 0. \tag{11}$$

Factoring a positive block matrix through a contraction shows that the dual optimum is (9). Strict primal feasibility follows by taking large scalar P, Q, x, y, so strong duality applies. This is the standard completely bounded trace-norm SDP [3], written with the realignment convention (8).

Factor the primal block matrix as $[F; G][F; G]^*$. Writing its columns as $\operatorname{vec}(A_s)$ and $\operatorname{vec}(\overline{B_s})$ gives $X = \sum_s A_s \otimes B_s$. The two partial traces are the column square sums in (6), up to transposes. Conversely, every such tensor factorization gives a feasible Gram block. Rescaling F and G reciprocally converts the minimum arithmetic mean into the geometric mean, proving (6).

For equal local dimensions define a map on matrices by

$$[\Phi_X(Y)]_{ab} = \sum_{cd} X_{(a,b),(c,d)} Y_{cd}. \tag{12}$$

Thus $\Phi_X(Y) = \sum_s A_s Y B_s^{\mathsf{T}}$, and its Choi matrix is $R(X)$. The SDP identifies $\mathcal{J}(X)$ with its completely bounded S_1 norm. Its Hilbert–Schmidt adjoint satisfies $\Phi_X^\dagger = \Phi_{X^*}$. Duality therefore identifies the completely bounded S_∞ norm of Φ_X with $\mathcal{J}(X^*)$. Schatten interpolation gives

$$\|X\| = \|\Phi_X\|_{S_2 \to S_2} \le \sqrt{\mathcal{J}(X)\mathcal{J}(X^*)}. \tag{13}$$

For unequal local dimensions, embed both spaces isometrically into a common dimension and pad X by zero. Both $\mathcal{J}$ norms and the operator norm are preserved, which proves the general assertion. □

Definition 3. The balanced factorization norm $\mathfrak{b}(X)$ is the infimum of the numbers b for which one decomposition $X = \sum_s A_s \otimes B_s$ satisfies

$$\sum_s A_s^* A_s, \quad \sum_s A_s A_s^*, \quad \sum_s B_s^* B_s, \quad \sum_s B_s B_s^* \ \preceq\ b I. \tag{14}$$

Each inequality acts on its own local space.

Concatenation and rescaling show that $\mathfrak{b}$ is subadditive and homogeneous. The tensor row-column Cauchy–Schwarz inequality also gives $\|X\| \le \mathfrak{b}(X)$. We need a common decomposition in (14); separately minimizing the row and column expressions does not suffice.

Lemma 4. *Suppose the local dimensions are equal, $V = V^*$ and $\mathcal{J}(V) \le 1$. There are factors $V = \sum_s A_s \otimes B_s$ such that the completely positive maps*

$$\Psi_A(Y) = \sum_s A_s Y A_s^*, \qquad \Psi_B(Y) = \sum_s B_s Y B_s^* \tag{15}$$

are Hilbert–Schmidt self-adjoint and have $S_2 \to S_2$ norm less than $3/2$.

Proof. By Lemma 2, $\Phi = \Phi_V$ is completely contractive at both S_1 and S_∞. Lemma 3.20 and Theorem 3.21 of Arhancet–Kriegler [2] give regular S_2 norm at most one and identify it with the decomposable norm. In finite matrix spaces, the latter is the infimum of the maximum $S_2 \to S_2$ norm of two completely positive diagonal maps in a completely positive block map

$$\mathcal{L}\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} = \begin{pmatrix} \Psi_A(Y_{11}) & \Phi(Y_{12}) \\ \Phi^\circ(Y_{21}) & \Psi_C(Y_{22}) \end{pmatrix}, \qquad \Phi^\circ(Y) = \Phi(Y^*)^*. \tag{16}$$

Choose such a completion with both diagonal norms below $3/2$.

Since $V = V^*$, $\Phi^\dagger = \Phi$. The Hilbert–Schmidt adjoint $\mathcal{L}^\dagger$ is completely positive, and $(\Phi^\circ)^\dagger = (\Phi^\dagger)^\circ$. Replacing $\mathcal{L}$ by $(\mathcal{L} + \mathcal{L}^\dagger)/2$ keeps the off-diagonal maps fixed and makes both diagonal maps Hilbert–Schmidt self-adjoint. Their S_2 norms remain below $3/2$.

Every Kraus operator for this block map is block diagonal. Indeed, an input supported in either diagonal block has zero positive output in the other block. The sum of positive Kraus contributions forces every cross block of every Kraus operator to vanish. Write the Kraus operators as $\operatorname{diag}(A_s, C_s)$. Then $\Phi(Y) = \sum_s A_s Y C_s^*$. Taking $B_s = \overline{C_s}$ gives the desired tensor representation. The map $\Psi_B(Y) = \overline{\Psi_C(\overline{Y})}$ has the same symmetry and norm. □

Hilbert–Schmidt self-adjointness of Φ is distinct from preservation of matrix adjoints. In (16), Φ° need not equal Φ. The imported theorem controls the diagonal maps at S_2; the operator-norm square-sum bounds will come from truncation.

3 Uniform approximate balancing

Theorem 5. *For every $0 < \delta \le 1$ and every finite-dimensional Hermitian V with $\mathcal{J}(V) \le 1$, there is a Hermitian W satisfying*

$$\|V - W\| \le \delta, \qquad \mathfrak{b}(W) \le F(\delta) \le C_0 \delta^{-4} [\log_2(2/\delta)]^5, \tag{17}$$

where $C_0 = 147 \cdot 2^{72}$ is universal. One explicit choice is

$$u = \log_2(2/\delta), \qquad L = \lceil 4u + 4\log_2 u + 64 \rceil, \tag{18}$$

$$F(\delta) = 4(2L+1)(2^{L+1} - 2^{-L}). \tag{19}$$

Proof. We first treat equal local dimensions and use the factors of Lemma 4. For $X = A, B$, define

$$R_X = \sum_{j=0}^{\infty} 2^{-j} \Psi_X^j(I). \tag{20}$$

The series converges in S_2, since $\|\Psi_X\|_{2\to 2}/2 < 3/4$. Every term is positive, $R_X \succeq I$, and self-adjointness gives

$$\Psi_X(R_X) = \Psi_X^\dagger(R_X) = 2(R_X - I) \preceq 2R_X. \tag{21}$$

The condition numbers of R_A, R_B can depend on dimension. Our truncation will use relative spectral bands rather than a bound on their largest eigenvalues.

Let P_j be the spectral projection of R_A for $[2^j, 2^{j+1})$, and let Q_j be the analogous projection for R_B. Empty projections are zero. There are finitely many nonzero projections. For $k, h \in \mathbb{Z}$, set

$$A_s^{[k]} = \sum_j P_{j+k} A_s P_j, \qquad B_s^{[h]} = \sum_j Q_{j+h} B_s Q_j, \tag{22}$$

$$V_{kh} = \sum_s A_s^{[k]} \otimes B_s^{[h]}. \tag{23}$$

Then $V = \sum_{k,h} V_{kh}$. The local bands obey

$$\sum_s A_s^{[k]} A_s^{[k]*} \preceq 4 \cdot 2^k I, \qquad \sum_s A_s^{[k]*} A_s^{[k]} \preceq 4 \cdot 2^{-k} I, \tag{24}$$

$$\sum_s B_s^{[h]} B_s^{[h]*} \preceq 4 \cdot 2^h I, \qquad \sum_s B_s^{[h]*} B_s^{[h]} \preceq 4 \cdot 2^{-h} I. \tag{25}$$

For example, the first sum is block diagonal in the output projections P_{j+k}. Its corresponding block is bounded by

$$\sum_s P_{j+k} A_s P_j A_s^* P_{j+k} \preceq 2^{-j} P_{j+k} \Psi_A(R_A) P_{j+k} \tag{26}$$

$$\preceq 2^{1-j} P_{j+k} R_A P_{j+k} \preceq 4 \cdot 2^k P_{j+k}. \tag{27}$$

For the column sum use $\Psi_A^\dagger(R_A) \preceq 2R_A$, the lower bound 2^{j+k} on the output block and the upper bound 2^{j+1} on the input block. The other two estimates are identical for B.

There is a second source of bounds. Define local unitaries

$$U_A(s) = \sum_j e^{ijs} P_j, \qquad U_B(t) = \sum_j e^{ijt} Q_j. \tag{28}$$

The operator V_{kh} is the (k, h) Fourier coefficient of the local conjugation of V by $U_A(s) \otimes U_B(t)$. Local-unitary invariance and the triangle inequality give

$$\mathcal{J}(V_{kh}) \le 1, \qquad \mathcal{J}(V_{kh}^*) \le 1. \tag{29}$$

The adjoint bound follows from the adjoint Fourier integral and $V = V^*$. Meanwhile, (6), (24) and (25) yield

$$\mathcal{J}(V_{kh}) \le 4 \cdot 2^{-(k+h)/2}, \qquad \mathcal{J}(V_{kh}^*) \le 4 \cdot 2^{(k+h)/2}. \tag{30}$$

Use the better endpoint bounds from (29) and (30) in (7) to obtain

$$\|V_{kh}\| \le 2 \cdot 2^{-|k+h|/4}. \tag{31}$$

In the other direction, the tensor Cauchy–Schwarz inequality gives

$$\|V_{kh}\| \le 4 \cdot 2^{-|k-h|/2}. \tag{32}$$

Indeed, pairing the A row bound with the B column bound gives $4 \cdot 2^{(k-h)/2}$, and reversing the two roles gives $4 \cdot 2^{(h-k)/2}$. Equation (31) controls bands moving in the same direction; (32) controls those moving in opposite directions.

Let $r = \max(|k|, |h|)$. Since $\max(|k+h|, |k-h|) \ge r$, the two bounds imply

$$\|V_{kh}\| \le 4 \cdot 2^{-r/4}. \tag{33}$$

Define $W_L = \sum_{|k|,|h|\le L} V_{kh}$. Adjunction reverses both Fourier indices, so $W_L = W_L^*$. There are $8r$ lattice points on the square shell $\max(|k|,|h|) = r$, and hence

$$\|V - W_L\| \le E_L := 32 \sum_{r=L+1}^{\infty} r 2^{-r/4} \tag{34}$$

$$= 32 a^{L+1} \left[\frac{L+1}{1-a} + \frac{a}{(1-a)^2} \right], \qquad a = 2^{-1/4}. \tag{35}$$

This sum is independent of the spectral range of either local weight.

For W_L concatenate the factors in (23) over the retained triples (s,k,h). Each of its four square sums is bounded by

$$B_L I, \qquad B_L = 4(2L+1) \sum_{k=-L}^{L} 2^k = 4(2L+1)(2^{L+1} - 2^{-L}) \le 8(2L+1)2^L. \tag{36}$$

For the A sums the factor $2L+1$ counts the choices of h; for B it counts k. Reversing the band index gives the same column bound.

It remains to check the explicit choice (18). Since $u \ge 1$, $a \le 7/8$ and $L \le 8u + 65 \le 73u$, we have

$$E_L \le 32 \cdot 2^{-L/4}(8L + 72) \le 32 \frac{\delta}{2u} 2^{-16} \cdot 656u = \frac{41}{256}\delta \le \delta. \tag{37}$$

Also $2^L \le 2^{69}\delta^{-4}u^4$ and $2L + 1 \le 147u$. Substitution in (36) proves (17).

For unequal local dimensions, pad V by local isometries into equal spaces. This preserves Hermiticity and $\mathcal{J}$. Apply the construction and compress its factors back to the original spaces. Compression contracts operator error and all four square-sum bounds: for instance, if P is the local support projection, $A^*PA \preceq A^*A$ and $APA^* \preceq AA^*$. Thus the same E_L, B_L apply. □

4 Compressing a balanced tensor decomposition

Lemma 6. *Suppose $W = \sum_j A_j \otimes B_j$ has all four square sums bounded by BI, where $B \ge 1$. For $0 < \epsilon \le 1$, set*

$$\zeta = \frac{\epsilon^2}{32B^2}, \qquad R = 2B \log \frac{2(d_A + d_B)}{\zeta}, \tag{38}$$

$$N = \left\lceil \left(\frac{64B^2}{\epsilon^2} + \frac{16R}{3\epsilon} \right) \log(4D) \right\rceil. \tag{39}$$

There is an operator $\widetilde{W}$ with $\mathrm{OSR}(\widetilde{W}) \le N$ and $\left\|W - \widetilde{W}\right\| \le \epsilon/2$.

Proof. Let g_j be independent standard circular complex Gaussian variables, with $\mathbb{E}|g_j|^2 = 1$, and define

$$A(g) = \sum_j g_j A_j, \qquad B(g) = \sum_j \overline{g_j} B_j, \qquad Z = A(g) \otimes B(g). \tag{40}$$

Each sample has operator Schmidt rank at most one, and $\mathbb{E}Z = W$. The complex Gaussian

fourth-moment identity gives

$$\mathbb{E}ZZ^* = WW^* + \left(\sum_j A_j A_j^*\right) \otimes \left(\sum_j B_j B_j^*\right), \tag{41}$$

$$\mathbb{E}Z^*Z = W^*W + \left(\sum_j A_j^* A_j\right) \otimes \left(\sum_j B_j^* B_j\right). \tag{42}$$

Since $\|W\| \le B$, both moments are bounded by $2B^2 I$.

The rectangular Gaussian matrix bound of Tropp [4, Theorem 1.5 and Corollary 4.2] applies to both local series. To include complex coefficients, write $g_j = (x_j + iy_j)/\sqrt{2}$ and use coefficients $A_j/\sqrt{2}$ and $iA_j/\sqrt{2}$. Their row and column variance sums remain bounded by BI. The same holds for the second factor. A union bound shows that the event

$$G = \{\|A(g)\| \le \sqrt{R},\ \|B(g)\| \le \sqrt{R}\} \tag{43}$$

has probability at least $1 - \zeta$. The two local factors are correlated; no independence between them is used in this union bound.

Clip the entire product sample, setting $\widehat{Z} = Z\mathbf{1}_G$. It has rank at most one, norm at most R, and both second moments remain bounded by $2B^2 I$. For every vector x, Cauchy–Schwarz gives

$$\|\mathbb{E}[Z\mathbf{1}_{G^c}]x\| \le \sqrt{\mathbb{P}(G^c)\mathbb{E}\,\|Zx\|^2} \le \sqrt{2B^2\zeta}\,\|x\| = \frac{\epsilon}{4}\,\|x\|. \tag{44}$$

The estimate controls the operator norm of the bias without replacing the matrix second moment by $\mathbb{E}\,\|Z\|^2$.

Take N independent clipped samples. After centering, each has norm at most $2R$ and both variance operators at most $2B^2 I$. Rectangular matrix Bernstein [4, Theorem 1.6], with $\tau = \epsilon/4$, yields

$$\mathbb{P}\left(\left\|\frac{1}{N}\sum_{j=1}^{N}\widehat{Z}_j - \mathbb{E}\widehat{Z}\right\| \ge \tau\right) \le 2D\exp\left[-\frac{N\tau^2}{4B^2 + (4/3)R\tau}\right] \le \frac{1}{2}. \tag{45}$$

The last inequality is exactly the choice (39). Choose a successful realization and let $\widetilde{W}$ be its mean. Its rank is at most N. Combining (44) and (45) gives the stated error. □

5 Proof of the dynamical bound

Proof of Theorem 1. Put $U = e^{-itH}$ and $M = e^{|t|}$. Corollary 6 of Ref. [1] gives

$$\mathcal{J}(U) \le M, \qquad \mathcal{J}(U^*) \le M. \tag{46}$$

For completeness, the same interface follows from (5): each interaction-picture $V(s)$ has $\mathcal{J}(V(s)) \le 1$, so its kth Dyson integral has $\mathcal{J}$ norm at most $|t|^k/k!$. The exact series converges absolutely in $\mathcal{J}$. The local free evolution preserves $\mathcal{J}$, and applying the argument at time $-t$ gives the adjoint bound. This argument requires no local frequency bound or time discretization.

Write $U = C + iS$, where $C = (U + U^*)/2$ and $S = (U - U^*)/(2i)$ are Hermitian. Then $\mathcal{J}(C/M), \mathcal{J}(S/M) \le 1$. Apply Theorem 5 to each, with $\delta = \epsilon/(4M)$. Multiplying the two approximants back by M and adding them with phase i gives W such that

$$\|U - W\| \le \epsilon/2, \qquad \mathfrak{b}(W) \le B := 2MF(\epsilon/(4M)). \tag{47}$$

The construction provides actual factors with this bound: concatenate the two decompositions, scale their factors by $\sqrt{M}$, and absorb i into one factor of the second decomposition. Lemma 6 then approximates W within $\epsilon/2$, proving the total error bound.

At fixed t, ϵ, B is independent of dimension, while $R = O(\log(2D))$. Equation (39) gives (3). For (4), set $q = 1 + T + \log(1/\epsilon)$. Equations (17) and (47) give

$$B \le C_1 e^{5T} \epsilon^{-4} q^5, \qquad \log(B/\epsilon) \le C_2 q \tag{48}$$

for universal constants. Use $d_A + d_B \le 2D$, $B/\epsilon \ge 1$ and $\log(2D) + q \ge 1$ in (39). Absorbing the ceiling into a universal constant proves (4). □

At fixed positive accuracy, $(\log(2D))^2 = o(D^c)$ for every $c > 0$. Theorem 1 therefore answers the positive-power question in Ref. [1] negatively. The lower-bound example in that reference is compatible with this result: it rules out constant rank and admits a logarithmic-dimensional approximation. The remaining logarithmic gap and the large constants in (4) are separate quantitative questions.

6 Checks and research record

The supplementary package contains two checkers. The local-band checker runs 374 assertions on complex rotations, band orientation, Fourier covariance, reconstruction, and a rational large-star reduction. Its floating-point matrix tests are finite controls, not all-dimensional certificates. Three invalid shortcuts are detected.

The direct-transfer checker runs 71 exact assertions. Finite quadrature verifies the Gaussian mean and both second moments in three matrix cases, including unequal local dimensions. Eight dyadic-parameter cases and symbolic calculations check the displayed constants. Four invalid substitutions are detected, including dropping WW^* and using column bounds alone for both variance operators. A hash manifest and offline runner accompany the code.

The work has received sequential internal proof and source review, not independent expert review or formal verification. The checked primary results distinguish the present endpoint statement from the spatially local compression theorems and the announced $\alpha < 1/2$ result. The full unpublished work cited as Ref. 93 of Ref. [1] was not available for comparison. No claim of worldwide first-publication priority is made. The author directed the research and preparation of the manuscript. AI assistance contributed to derivation, literature comparison, implementation, checking, and drafting.

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