

Classical counterexamples to monotonicity of noninteger Rényi reflected entropy

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Abstract

For every noninteger order $\alpha > 1$, we construct a finite classical state whose Rényi reflected entropy increases after discarding part of the second subsystem. The construction uses $n = \lfloor \alpha \rfloor + 2$ values of the first variable, $2n$ values of the second, and a binary variable that is discarded. A negative entry of a noninteger matrix power supplies the derivative of the reverse inequality; an explicit probability distribution realizes the required Gram-matrix perturbation. All parameters can be chosen rational when α is rational. At $\alpha = 5/2$, a twelve-entry rational distribution on $4 \times 8 \times 2$ values gives a reverse entropy gap greater than 6×10^{-8} in natural logarithms. This example is certified both by rational interval arithmetic and by a separate verified eigensystem calculation.

1 The question and the result

The canonical purification of a bipartite density operator ρ_{AX} is $|\sqrt{\rho_{AX}}\rangle_{AXA^*X^*}$, where the starred systems are conjugate copies in the fixed defining basis. Write

$$\sigma_{AA^*}(\rho_{AX}) = \text{Tr}_{XX^*} |\sqrt{\rho_{AX}}\rangle \langle \sqrt{\rho_{AX}}|, \quad S_R^{(\alpha)}(A : X)_\rho = \frac{\log \text{Tr} \sigma_{AA^*}(\rho_{AX})^\alpha}{1 - \alpha}. \quad (1.1)$$

A necessary property of a correlation measure is that discarding a subsystem cannot increase it. For reflected entropy, this asks whether

$$S_R^{(\alpha)}(A : BC)_\rho \geq S_R^{(\alpha)}(A : B)_{\text{Tr}_C \rho}. \quad (1.2)$$

Dutta and Faulkner proved subsystem monotonicity at integer orders $\alpha \geq 2$ using a replica argument [4, Appendix B]. Hayden, Lemm, and Sorce exhibited classical counterexamples for $0 < \alpha < 2$, including the von Neumann limit [1]. Their examples do not settle finite noninteger orders above two. The min-reflected-entropy limit is a separate result [3]; taking that limit does not settle any fixed finite order.

Theorem 1.1. *For every noninteger $\alpha > 1$, there exists a diagonal density operator on $\mathbb{C}^n \otimes \mathbb{C}^{2n} \otimes \mathbb{C}^2$, with $n = \lfloor \alpha \rfloor + 2$, such that*

$$S_R^{(\alpha)}(A : B)_{\text{Tr}_C \rho} - S_R^{(\alpha)}(A : BC)_\rho > 0. \quad (1.3)$$

If α is rational, the state can be chosen with rational entries by the explicit formulas in Section 3.

The matrix-analysis mechanism used below is the failure of entrywise positivity for certain noninteger conventional powers of doubly nonnegative matrices. This phenomenon, including the critical exponent, belongs to the existing theory [2]. The step needed here is to realize the pertinent matrix perturbation by one state and its ordinary partial trace. Positivity of arbitrary Gram matrices alone would not establish that connection.

2 Classical reflected states and a realizable perturbation

Let $\rho_{AX} = \sum_{a,x} p(a,x) |a,x\rangle\langle a,x|$ be diagonal. Its canonical purification is

$$|\sqrt{\rho_{AX}}\rangle = \sum_{a,x} \sqrt{p(a,x)} |a,x\rangle |a,x\rangle.$$

Tracing XX^* gives an operator supported on the span of $|a,a\rangle$. In that basis it is the Gram matrix

$$G(p)_{aa'} = \sum_x \sqrt{p(a,x)p(a',x)}. \quad (2.1)$$

All other eigenvalues of σ_{AA^*} are zero. Thus its positive-order spectral traces are exactly those of $G(p)$.

Lemma 2.1. *Let T be the adjacency matrix of a path with vertices $1, \dots, n$, let $0 < e < 1/2$, and let $0 < t < (1 - e)/n$. Put*

$$G = \frac{I + eT}{n}, \quad H = G + t(E_{1n} + E_{n1}).$$

There is a classical state on $n \times 2n \times 2$ values whose reflected matrices for $A : BC$ and $A : B$ are G and H , respectively.

Proof. Use three disjoint types of labels for B . For each of the $n - 1$ path edges $(i, i + 1)$, assign one label b_i , and set

$$p(i, b_i, 0) = p(i + 1, b_i, 0) = e/n.$$

For a further label b_* , set

$$p(1, b_*, 0) = t, \quad p(n, b_*, 1) = t.$$

Finally, assign a private label $\tilde{b}_i$ to each vertex and set

$$p(i, \tilde{b}_i, 0) = \frac{1 - \deg_T(i)e}{n} - t \mathbf{1}_{\{1,n\}}(i). \quad (2.2)$$

All unspecified probabilities vanish. The interior private probabilities are positive because $2e < 1$, and the endpoint probabilities are positive by the bound on t . The marginal probability of each A value is $1/n$, so normalization follows.

In (2.1) with $x = (b, c)$, each edge label gives the corresponding off-diagonal entry e/n . The two endpoint contributions at b_* have distinct C values and give no off-diagonal entry. Hence the matrix is G . After summing over C , both contributions share b_* , producing exactly t in positions $(1, n)$ and $(n, 1)$. No other entry changes. The second matrix is therefore H . $\square$

For comparison, discarding a classical variable always increases the Gram matrix entrywise: for each fixed b , the Cauchy–Schwarz inequality gives

$$\sum_c \sqrt{p(a, b, c)p(a', b, c)} \leq \sqrt{\sum_c p(a, b, c) \sum_c p(a', b, c)}.$$

For an integer $r \geq 2$, the polynomial $\text{Tr } G^r$ has nonnegative coefficients in the nonnegative matrix entries, and hence $\text{Tr } H^r \geq \text{Tr } G^r$. The sign of the entropy prefactor then gives (1.2) for classical states at integer orders. The construction below exploits precisely the loss of this entrywise polynomial argument at noninteger orders. The corresponding statement for general quantum states follows from the integer-replica argument of [4, Appendix B].

3 Explicit parameters for every noninteger order

Fix a noninteger $\alpha > 1$ and define

$$\beta = \alpha - 1, \quad k = \lfloor \beta \rfloor + 2, \quad n = k + 1, \quad e = 2^{-k-3}, \quad (3.1)$$

$$c = \binom{\beta}{k}, \quad d_0 = n^{-\lceil \beta \rceil} |c| e^k, \quad L_0 = 8n \lceil \alpha \rceil^2, \quad t = \min \left\{ \frac{1-2e}{2n}, \frac{d_0}{2L_0} \right\}. \quad (3.2)$$

The final factor in $\beta(\beta-1)\cdots(\beta-k+1)$ is negative and every earlier factor is positive. In particular, $c < 0$ and $d_0, t > 0$.

Lemma 3.1. *For the path matrix T on $n = k + 1$ vertices,*

$$((I + eT)^\beta)_{1n} \leq -\frac{|c|e^k}{2}. \quad (3.3)$$

Proof. Since $\|T\| \leq 2$ and $2e < 1$, the binomial series converges absolutely in operator norm. The distance between vertices 1 and n is k , so $(T^j)_{1n} = 0$ for $j < k$ and $(T^k)_{1n} = 1$. Bipartiteness also gives $(T^{k+1})_{1n} = 0$. For $j \geq k$,

$$\left| \frac{\binom{\beta}{j+1}}{\binom{\beta}{j}} \right| = \frac{j - \beta}{j + 1} < 1.$$

Consequently

$$\left| ((I + eT)^\beta)_{1n} - ce^k \right| \leq |c| \sum_{j=k+2}^{\infty} (2e)^j = |c| \frac{(2e)^{k+2}}{1-2e}.$$

Relative to $|c|e^k$, this is at most

$$\frac{2^{k+2}e^2}{1-2e} = \frac{2^{-k-4}}{1-2^{-k-2}} < \frac{1}{2},$$

as $k \geq 2$. This proves (3.3). $\square$

Proof of Theorem 1.1. Set $B = E_{1n} + E_{n1}$ and $f(u) = \text{Tr}(G + uB)^\alpha$, with $G = (I + eT)/n$. By spectral differentiation and Lemma 3.1,

$$f'(0) = 2\alpha(G^\beta)_{1n} \leq -\alpha n^{-\beta} |c| e^k \leq -d_0. \quad (3.4)$$

For $0 \leq u \leq t$, Weyl's bound and $\|B\| = 1$ give

$$\lambda_{\min}(G + uB) \geq \frac{1-2e}{n} - t \geq \frac{1-2e}{2n} \geq \frac{1}{4n}.$$

These matrices have trace one and are positive, so their spectra lie in $[1/(4n), 1]$.

For completeness, in an eigenbasis of $G + uB$ the Hessian of the spectral trace is

$$f''(u) = \alpha \sum_{i,j} \frac{\lambda_i^{\alpha-1} - \lambda_j^{\alpha-1}}{\lambda_i - \lambda_j} |B_{ij}|^2,$$

where the diagonal quotient is its continuous limit. The mean-value theorem and $\|B\|_F^2 = 2$ imply

$$0 \leq f''(u) \leq 2\alpha(\alpha-1) \max_{x \in [1/(4n), 1]} x^{\alpha-2} \leq L_0.$$

For $\alpha \geq 2$ the maximum is one; for $1 < \alpha < 2$ it is at most $4n$. Thus the same stated L_0 covers both cases. Taylor's theorem, (3.4), and $t \leq d_0/(2L_0)$ now give

$$\text{Tr } G^\alpha - \text{Tr}(G + tB)^\alpha \geq d_0 t - \frac{L_0 t^2}{2} \geq \frac{3}{4} d_0 t > 0. \quad (3.5)$$

The choice of t satisfies the positivity requirement of Lemma 2.1, so G and $G + tB$ are the reflected matrices of its actual state and partial trace. Because $(1 - \alpha)^{-1} < 0$, (3.5) proves the reversed entropy inequality. More explicitly, using $\log(x/y) \geq (x - y)/x$ for $x > y > 0$ and $\text{Tr } G^\alpha \leq 1$ yields

$$S_R^{(\alpha)}(A : B) - S_R^{(\alpha)}(A : BC) \geq \frac{3d_0 t}{4(\alpha - 1)}. \quad (3.6)$$

The dimension is $n = \lfloor \alpha \rfloor + 2$. For rational noninteger α , every parameter in (3.2) and every probability in (2.2) is rational. $\square$

The bound in (3.6) is conservative and is not uniform near integer orders. Neither a common state for all orders nor a fixed dimension for unbounded α is claimed.

4 A twelve-entry witness at order five halves

The following larger-gap witness is convenient for independent checking. Number A from 0 to 3, B from 0 to 7, and C from 0 to 1. Table 1 lists every nonzero probability.

Triples (a, b, c)	Probability at each triple
$(0, 0, 0), (3, 0, 1)$	$1/40000$
$(0, 1, 0), (1, 1, 0), (1, 2, 0),$ $(2, 2, 0), (2, 3, 0), (3, 3, 0)$	$1/16$
$(0, 4, 0), (3, 7, 0)$	$7499/40000$
$(1, 5, 0), (2, 6, 0)$	$1/8$

Table 1: A rational distribution on $4 \times 8 \times 2$ values.

All entries are positive and their sum is $2/40000 + 6/16 + 2(7499/40000) + 2/8 = 1$. Define

$$G(d) = \frac{1}{4} \begin{pmatrix} 1 & e & 0 & d \\ e & 1 & e & 0 \\ 0 & e & 1 & e \\ d & 0 & e & 1 \end{pmatrix}, \quad e = \frac{1}{4}.$$

The reflected matrices are $G(0)$ for $A : BC$ and $G(10^{-4})$ for $A : B$. Reflection of the coordinate order splits $4G(d)$ into

$$\begin{pmatrix} 1+d & e \\ e & 1+e \end{pmatrix}, \quad \begin{pmatrix} 1-d & e \\ e & 1-e \end{pmatrix}.$$

With

$$a(d) = \frac{e+d}{2}, \quad b(d) = \frac{\sqrt{(e-d)^2 + 4e^2}}{2},$$

its four eigenvalues are

$$\lambda_{s,t}(d) = \frac{1 + s a(d) + t b(d)}{4}, \quad s, t \in \{1, -1\}. \quad (4.1)$$

They are strictly positive at both values of d .

Proposition 4.1. *For the state in Table 1,*

$$\frac{14}{10^9} < \text{Tr } G(0)^{5/2} - \text{Tr } G(10^{-4})^{5/2} < \frac{15}{10^9}, \quad (4.2)$$

$$S_R^{(5/2)}(A : B) - S_R^{(5/2)}(A : BC) > \frac{6}{10^8}. \quad (4.3)$$

Proof. The supplied rational certificate encloses each square root in (4.1) between rational endpoints whose squares bracket its rational argument. It evaluates $\lambda^{5/2} = \lambda^2 \sqrt{\lambda}$ by outward rational arithmetic and subtracts the two enclosing sums. The resulting interval is strictly inside $(14, 15) \times 10^{-9}$. The same computation certifies

$$\frac{2}{3} \frac{\text{Tr } G(0)^{5/2} - \text{Tr } G(10^{-4})^{5/2}}{\text{Tr } G(0)^{5/2}} > 6 \times 10^{-8}.$$

Applying $\log(x/y) \geq (x-y)/x$ proves the entropy bound without a numerical logarithm. $\square$

A second calculation reconstructs the two matrices from the probability table and uses verified eigensystems at 384-bit Arb precision. It does not use the rational checker or formula (4.1). Its enclosing balls give

$$\begin{aligned} \text{Tr } G(0)^{5/2} - \text{Tr } G(10^{-4})^{5/2} &= 1.4666200625084441957 \dots \times 10^{-8}, \\ S_R^{(5/2)}(A : B) - S_R^{(5/2)}(A : BC) &= 6.6556627612590249318 \dots \times 10^{-8}. \end{aligned}$$

The respective ball radii are below 3.4×10^{-115} and 2.0×10^{-114} . At orders 2 and 3, the trace differences are instead -1.25×10^{-9} and -9.375×10^{-10} , consistent with the classical integer-order argument.

5 Reproducibility and scope

The companion evidence archive contains the original probability table, rational checker, the independent Arb checker and its output, and a hash manifest linking this manuscript to candidate M10-v4. These finite computations certify Proposition 4.1; Theorem 1.1 follows from the analytic argument and does not rely on sampling orders.

The theorem disproves subsystem monotonicity for every finite noninteger order above two, already on diagonal states. It also supplies another construction in the previously studied interval $(1, 2)$. Integer orders obey the known quantum monotonicity inequality and are excluded from this construction. No dimension-optimality of the witness is claimed. The primary-source comparison identifies the inherited matrix-power mechanism and the earlier reflected-entropy counterexamples; it is not an exhaustive novelty clearance. The independent calculation refers to a separate implementation and arithmetic route, not to external expert review.

AI-assisted research

This work was developed in an author-directed, AI-assisted research workflow. AI systems assisted with mathematical derivation, literature checks, computational verification, and manuscript drafting. The internal reviews and separate implementations described above were produced within this workflow and do not constitute external expert review.

References

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