

Logarithmic mixing of a Gaussian KMS sampler for weakly interacting hopping fermions

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Version 1
22 October 2026

Based on candidate M07-v1. Proof and primary-source comparisons received separate-context internal AI review; computational checks were replayed. No external expert review or formal verification is claimed.

Abstract

We prove a volume-independent weak-interaction regime for logarithmic mixing of the Gaussian KMS-detailed-balance sampler with physical fermionic creation and annihilation couplings. The reference Hamiltonian may have arbitrary bounded finite-range Hermitian hopping on a lattice of fixed dimension $D \geq 2$. At every fixed inverse temperature, sufficiently weak bounded finite-range even interactions preserve an $O(\log(N/\varepsilon))$ mixing bound for even initial states. The proof uses physical-site commutator gradients, an operator-norm-contracting parity twist, and an integrable spatial kernel for the free gradient evolution. We also establish local continuity of the prescribed coherent correction, using a positive integrable kernel for its hyperbolic-tangent multiplier. These estimates close a Volterra inequality without a volume-dependent perturbation norm.

1 The sampler and the theorem

Fix $\beta > 0$ and a lattice dimension $D \geq 2$. Let Λ range over finite boxes with N fermionic modes and $|B(i, r)| \leq C_b(1 + r)^D$. Consider

$$H_\lambda = H_0 + \lambda V, \quad H_0 = \sum_{i,j} M_{ij} c_i^\dagger c_j, \quad V = \sum_X v_X. \quad (1.1)$$

Here M is Hermitian, has fixed finite range, and has uniformly bounded local strength; in particular $\|M\| \leq J$. The interactions v_X are Hermitian, parity-even, of fixed finite range and uniformly bounded norm and overlap. All constants below depend on this bounded class and on β , not on N or on an individual instance.

Use exactly the Gaussian filter of Šmíd et al. [1, Eqs. (17)–(20)]:

$$\widehat{f}(u) = \exp \left[-\frac{(\beta u + 1)^2}{8} + \frac{1}{8} \right], \quad (1.2)$$

$$f(t) = \sqrt{\frac{2}{\pi\beta^2}} \exp \left[-\frac{2}{\beta^2} \left(t - \frac{i\beta}{4} \right)^2 \right]. \quad (1.3)$$

For the physical list $A_{i,-} = c_i$, $A_{i,+} = c_i^\dagger$, set

$$K_{i,\sigma}(\lambda) = \widehat{f}(\text{ad}_{H_\lambda})A_{i,\sigma} = \int_{\mathbb{R}} f(t)\tau_t^\lambda(A_{i,\sigma})dt, \quad (1.4)$$

$$B_i(\lambda) = \sum_{\sigma=\pm} K_{i,\sigma}^\dagger K_{i,\sigma}, \quad G_i(\lambda) = \frac{i}{2} \tanh\left(\frac{\beta \text{ad}_{H_\lambda}}{4}\right) B_i(\lambda). \quad (1.5)$$

The Heisenberg generator is

$$\mathcal{L}_\lambda(O) = \sum_{i,\sigma} \left(K_{i,\sigma}^\dagger O K_{i,\sigma} - \frac{1}{2} \{K_{i,\sigma}^\dagger K_{i,\sigma}, O\} \right) + i \left[\sum_i G_i(\lambda), O \right]. \quad (1.6)$$

The sum is unrescaled. In the source's time-kernel representation, the optional frequency cutoff equals one on every finite-volume Bohr frequency, so (1.5) is the exact same coherent completion.

Although the source construction is stated for self-adjoint couplings, the pair $(c_i, c_i^\dagger)$ is unitarily equivalent as a label list to

$$(c_i + c_i^\dagger)/\sqrt{2}, \quad i(c_i - c_i^\dagger)/\sqrt{2}.$$

The transformation preserves both the sum of dissipators and $\sum K^\dagger K$, hence also the coherent correction. No rate factor is introduced. The source KMS construction therefore fixes $\rho_{\beta,\lambda} = e^{-\beta H_\lambda} / \text{Tr } e^{-\beta H_\lambda}$.

Theorem 1.1. *For the bounded class above there are $\lambda_*(\beta) > 0$ and $C_\beta < \infty$ such that for all $|\lambda| \leq \lambda_*(\beta)$,*

$$\sup_{\rho \text{ even}} \frac{1}{2} \left\| e^{t\mathcal{L}_\lambda^*}(\rho) - \rho_{\beta,\lambda} \right\|_1 \leq \varepsilon \quad \text{if} \quad t \geq C_\beta \log(N/\varepsilon), \quad 0 < \varepsilon < 1. \quad (1.7)$$

The Gibbs density is the unique even stationary density.

The perturbative fermion theorem in [1] treats a non-hopping reference, whereas diagonalizing a hopping matrix generally makes the interactions nonlocal in its eigenmodes. Recent one-dimensional results give $O(\log^4 N + \log(1/\varepsilon))$ mixing for a KMS sampler with single-site Pauli couplings [4] and logarithmic mixing for constant-block Petz heat-bath dynamics [5]. For weakly interacting fermions in arbitrary fixed dimension, Han, Zhao, and Liu establish Gaussian-mixture decompositions and a polynomial-time classical sampler [6]. These results concern different geometries, generators, or sampling outputs. Here we prove (1.7) for the prescribed physical-jump Lindbladian in the physical lattice. The integrable spatial kernel below provides the trace-norm control that a spectral gap alone would not supply.

2 Free gradients and the parity twist

Let P denote total parity and define $\mathcal{J}(X) = PX$. This is an isometric involution in operator norm. For a Heisenberg generator $\mathcal{L}$, write

$$\mathcal{L}^{\text{tw}} = \mathcal{J}\mathcal{L}\mathcal{J}. \quad (2.1)$$

Since $e^{t\mathcal{L}^{\text{tw}}} = \mathcal{J}e^{t\mathcal{L}}\mathcal{J}$, the twisted semigroup is contractive whenever $e^{t\mathcal{L}}$ is unital completely positive. The twisted map need not itself be positive.

For even O , introduce the odd physical-site gradients

$$\partial_i O = [c_i, O], \quad \bar{\partial}_i O = [c_i^\dagger, O]. \quad (2.2)$$

If ℓ is a linear odd combination of creation and annihilation operators, direct CAR expansion gives

$$\partial_i \mathcal{D}(\ell) O = \mathcal{D}(\ell)^{\text{tw}} \partial_i O - \frac{1}{2} (\{c_i, \ell^\dagger\} [\ell, O] + \{c_i, \ell\} [\ell^\dagger, O]). \quad (2.3)$$

Both anticommutators in the last expression are scalars. To see the twist, for an odd ℓ the recycling term in $\mathcal{J}\mathcal{D}(\ell)\mathcal{J}$ changes sign whereas the two loss terms do not. Expanding the left side of (2.3) and moving c_i through $\ell, \ell^\dagger$ then leaves exactly the two displayed scalar terms. The creation-gradient identity follows by replacing c_i with $c_i^\dagger$.

Put $F_- = \hat{f}(-M)$ and $F_+ = \hat{f}(M)$. For complex Hermitian hopping the annihilation coefficient matrix is F_- and the creation coefficient matrix is $F_+^\top$. Thus

$$R = F_-^2 + F_+^2 = 2e^{-\beta^2 M^2/4} \cosh(\beta M/2) \geq r_0 I, \quad r_0 = 2e^{-\beta^2 J^2/4} > 0. \quad (2.4)$$

The total free loss is

$$\sum_i B_i(0) = \text{Tr}(F_+^2) I + c^\dagger (F_-^2 - F_+^2) c.$$

It commutes with H_0 , so $\sum_i G_i(0) = 0$. Individual $G_i(0)$ need not vanish.

Summing (2.3) over the actual physical coupling list yields

$$\partial \mathcal{L}_0 O = \mathcal{L}_0^{\text{tw}} \partial O - \frac{1}{2} R \partial O, \quad \bar{\partial} \mathcal{L}_0 O = \mathcal{L}_0^{\text{tw}} \bar{\partial} O - \frac{1}{2} R^\top \bar{\partial} O. \quad (2.5)$$

Here the gradients are column vectors indexed by physical sites and the twisted generator acts componentwise. The transpose in the second identity is needed for complex hopping.

3 An integrable spatial kernel

For $\mu > 0$, define the weighted matrix norm

$$\|A\|_\mu = \max_i \sum_j e^{\mu \text{dist}(i,j)} |A_{ij}|. \quad (3.1)$$

The triangle inequality for the distance makes it submultiplicative. Finite range and local bounds give $\|M\|_\mu \leq J_\mu$ uniformly in N . Using absolute values in the entire power series for (2.4) gives

$$\|R\|_\mu \leq b_\mu := 2e^{\beta^2 J_\mu^2/4} \cosh(\beta J_\mu/2). \quad (3.2)$$

Lemma 3.1. *There is a uniform nonnegative function $\mathsf{K}(t)$ such that both row and column sums of $|e^{-tR/2}|$ are at most $\mathsf{K}(t)$ and*

$$\mathsf{K}(t) = C_\beta (1+t)^D e^{-r_0 t/2} \quad (3.3)$$

is a valid choice after increasing C_β . For $\gamma = r_0/4$,

$$A_\beta = \int_0^\infty e^{\gamma t} \mathsf{K}(t) dt < \infty, \quad B_\beta = \sup_{t \geq 0} e^{\gamma t} \mathsf{K}(t) < \infty. \quad (3.4)$$

Proof. Let $S_t = e^{-tR/2}$. Equation (3.2) gives $\|S_t\|_\mu \leq e^{b_\mu t/2}$, while (2.4) gives the spectral norm bound $\|S_t\| \leq e^{-r_0 t/2}$. In a fixed row, split the sites into $\text{dist}(i, j) \leq vt$ and its complement. The near sum is at most $C_b (1+vt)^D e^{-r_0 t/2}$, since each entry is bounded by the spectral norm. The far sum is at most $e^{-\mu vt} e^{b_\mu t/2}$. Choose $\mu v \geq (b_\mu + r_0)/2$ to obtain (3.3). Hermiticity of S_t makes its absolute row and column sums equal at each site; transposing R yields the same bounds. The integrability assertions follow because $\gamma < r_0/2$. $\square$

This estimate allows finite transient growth of physical-site gradients. Requiring instantaneous contraction in their unweighted sum is unnecessary; the integrable kernel is what will enter the perturbation argument.

4 Locality of the full interacting sampler

4.1 Filtered jumps and products

The fermionic Lieb–Robinson bound for finite-range even interactions [2] implies uniform local tails for $\tau_t^\lambda(A_i)$, for $|\lambda| \leq 1$. For an odd filtered jump, choose a local Majorana γ_i and localize the even operator $K_{i,\sigma}\gamma_i$ before multiplying back by γ_i . The exterior-even test uses a commutator and the exterior-odd test uses the graded anticommutator. This supplies a contractive local approximation $K_{i,\sigma}^{(r)}$ on $B(i, r)$ without applying an even-only localization statement to an odd operator.

Splitting the Gaussian filter integral at time proportional to r gives

$$\|K_{i,\sigma} - K_{i,\sigma}^{(r)}\| \leq Ce^{-cr}. \quad (4.1)$$

All jump norms are also uniformly bounded. Local Duhamel and the same Lieb–Robinson estimate give

$$\|K_{i,\sigma}(\lambda) - K_{i,\sigma}(0)\| \leq C|\lambda|Te^{vT} + 2 \int_{|t|>T} |f(t)| dt. \quad (4.2)$$

The right side tends to zero by first choosing T large and then λ small. This is uniform in site and volume.

We will need convergence in a local norm, not just in operator norm. If an observable about i is written as shell terms $A = \sum_{r \geq 0} A_r$, supported on $B(i, r)$, define

$$\|A\|_{a,\text{loc}} = \sum_{r \geq 0} e^{ar} \|A_r\|.$$

The particular shell decomposition obtained by consecutive conditional expectations suffices. Bounds (4.1) and (4.2) give shell differences bounded by $C \min\{\eta_0(\lambda), e^{-cr}\}$ with $\eta_0(\lambda) \rightarrow 0$. A finite-radius split therefore proves convergence in every weaker norm with $0 < a < c$. Products obey the same property after weakening the exponent if necessary. In particular $B_i(\lambda)$ has uniform exponential tails and $B_i(\lambda) - B_i(0) \rightarrow 0$ in a weaker local norm.

4.2 The coherent correction

The hyperbolic-tangent multiplier can be represented without an absolutely nonintegrable $1/t$ kernel. Define

$$p_\beta(t) = \frac{4}{\pi\beta} \log \coth\left(\frac{\pi|t|}{\beta}\right). \quad (4.3)$$

Lemma 4.1. *The kernel p_β is nonnegative, integrable, and normalized. Its Fourier transform satisfies*

$$\int_{\mathbb{R}} p_\beta(t) e^{itu} dt = \frac{\tanh(\beta u/4)}{\beta u/4}, \quad (4.4)$$

with value one at $u = 0$. Consequently the exact coherent term is

$$G_i(\lambda) = \frac{\beta}{8} \int_{\mathbb{R}} p_\beta(t) \tau_t^\lambda(i[H_\lambda, B_i(\lambda)]) dt. \quad (4.5)$$

Proof. For $x > 0$, $\log \coth x = 2 \sum_{n \geq 1, n \text{ odd}} e^{-2nx}/n$. Termwise integration of this nonnegative expansion shows integrability and $\int p_\beta = 8\pi^{-2} \sum_{n \text{ odd}} n^{-2} = 1$. Fourier transforming the exponential terms, or equivalently summing their partial fractions, gives

$$\hat{p}_\beta(u) = \frac{8}{\pi^2} \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \frac{1}{n^2 + (\beta u/2\pi)^2} = \frac{\tanh(\beta u/4)}{\beta u/4}.$$

This identity also appears in the quantum-belief-propagation formulation of [3, Lemma 3.1]. The kernel has an integrable logarithmic singularity at zero and exponentially decaying tails. On a Bohr-frequency component with frequency u , multiplying (4.4) by $i\beta u/8$ produces $(i/2) \tanh(\beta u/4)$, including its zero-frequency limit. Applying finite-volume spectral calculus proves (4.5). $\square$

Lemma 4.2. *The operators $G_i(\lambda)$ have uniformly bounded exponential local tails for $|\lambda| \leq 1$, and*

$$\sup_i \|G_i(\lambda) - G_i(0)\| \longrightarrow 0$$

uniformly in volume. The difference also converges in a weaker exponential local norm.

Proof. For a shell A_r supported in $B(i, r)$, only interaction terms meeting that ball contribute to $[H_\lambda, A_r]$. Their summed norm is at most a fixed constant times $(1+r)^D \|A_r\|$, and the support grows by at most the Hamiltonian range. Thus commutation with H_λ maps an exponentially local shell expansion to another such expansion at any sufficiently weaker exponent, with uniform constants.

Apply this to $C_i(\lambda) = i[H_\lambda, B_i(\lambda)]$. The preceding product estimates give uniform tails. Its difference is

$$C_i(\lambda) - C_i(0) = i[H_0, B_i(\lambda) - B_i(0)] + i\lambda[V, B_i(\lambda)].$$

The first term tends to zero in a weaker local norm, and the second is uniformly $O(|\lambda|)$ in another weaker local norm, by the same shell-counting estimate.

To localize (4.5) to radius r , split the shell expansion of C_i at radius proportional to r and the time integral at $|t| = \theta r$. The discarded spatial shells and the large-time part are exponentially small. On the remaining shells and times, choose θ small enough for the Lieb–Robinson spatial decay to dominate its time growth. This proves the uniform exponential tail for G_i .

For continuity, first restrict the integral to a fixed time interval and C_i to a fixed ball. The change in C_i tends to zero, and Duhamel bounds the change of the finite-time local evolution by a constant times $|\lambda|$. The uniform spatial and time tails make the discarded contribution arbitrarily small. Taking these cutoffs first and then $\lambda \rightarrow 0$ proves operator-norm continuity. A final fixed-radius shell split upgrades it to convergence in a weaker exponential local norm. $\square$

In particular, the full coherent completion is retained. This argument supplies the hopping-reference estimate directly; it does not import a perturbation lemma whose reference Hamiltonian is assumed non-hopping.

4.3 Local superoperator shells

Group the generator by site, including $i[G_i, \cdot]$, and let $\mathcal{E}_i = \mathcal{L}_i(\lambda) - \mathcal{L}_i(0)$. Use the same conditional-expectation approximants at λ and at zero, take the difference of their local generators, and then telescope over the radius. This gives

$$\mathcal{E}_i = \sum_{r \geq 0} \mathcal{E}_{i,r}, \quad X_{i,r} = B(i, r), \quad \mathcal{E}_{i,r}(I) = 0, \quad (4.6)$$

where every shell is a local, total-even superoperator. The previous lemmas and the elementary dissipator difference bound give

$$\|\mathcal{E}_{i,r}\| \leq \epsilon_r(\lambda), \quad \epsilon_0 = C\eta(\lambda), \quad \epsilon_r = C \min\{\eta(\lambda), e^{-cr}\} \quad (r \geq 1), \quad (4.7)$$

with $\eta(\lambda) \rightarrow 0$. These are norms induced by the full operator norm, not only its restriction to even observables. Therefore

$$\kappa(\lambda) = 6C_b^2 \sum_{r \geq 0} (1+r)^{2D} \epsilon_r(\lambda) \longrightarrow 0 \quad (4.8)$$

uniformly.

5 Gradient stability

For even O set

$$\mathcal{S}(O) = \sum_i \left(\|\partial_i O\| + \|\bar{\partial}_i O\| \right). \quad (5.1)$$

Let $\mathcal{T}_i$ be tracial expectation onto the complement of site i . On even observables it can be expressed as the average of conjugation by the identity and the three local CAR Pauli unitaries. If $\gamma_1 = c_i + c_i^\dagger$, $\gamma_2 = i(c_i - c_i^\dagger)$, and $Z_i = 1 - 2c_i^\dagger c_i$, then $\|[Z_i, O]\| \leq \|[\gamma_1, O]\| + \|[\gamma_2, O]\|$. The averaging formula consequently gives

$$\|O - \mathcal{T}_i O\| \leq \|[c_i, O]\| + \|[c_i^\dagger, O]\|.$$

Telescoping the commuting expectations yields

$$\|O - \text{tr}(O)I\| \leq \mathcal{S}(O), \quad \mathcal{S}(O) \leq 4N \|O\|, \quad \text{tr}(O) = 2^{-N} \text{Tr } O. \quad (5.2)$$

Lemma 5.1. *For a local total-even superoperator $\mathcal{E}_X$ satisfying $\mathcal{E}_X(I) = 0$ and even O ,*

$$\partial_i \mathcal{E}_X O - \mathcal{E}_X^{\text{tw}} \partial_i O = 0 \quad (i \notin X). \quad (5.3)$$

For $i \in X$, the norm of this difference is at most

$$3 \|\mathcal{E}_X\| \sum_{j \in X} (\|\partial_j O\| + \|\bar{\partial}_j O\|).$$

The same assertions hold for $\bar{\partial}_i$.

Proof. Expand $\mathcal{E}_X$ into terms pOq with local coefficients of matching parity. For $i \notin X$, moving c_i past both coefficients gives the sign $(-1)^{|p|}$; this is exactly the sign from $\mathcal{J}(p\mathcal{J}(\cdot)q)$. This proves (5.3) term by term.

The even observable $\mathcal{T}_X O$ commutes with the local coefficients, so $\mathcal{E}_X \mathcal{T}_X O = \mathcal{T}_X O \mathcal{E}_X(I) = 0$. By telescoping, $\|\mathcal{E}_X O\| \leq \|\mathcal{E}_X\| \sum_{j \in X} (\|\partial_j O\| + \|\bar{\partial}_j O\|)$. Applying ∂_i , whose norm is at most two, bounds the first term of the forcing. The second costs at most $\|\mathcal{E}_X^{\text{tw}}\| \|\partial_i O\| = \|\mathcal{E}_X\| \|\partial_i O\|$. Adding gives the factor three. $\square$

Let $O(t) = e^{t\mathcal{L}} O$. Combine the free identity (2.5) with Lemma 5.1, and move the *full* $\mathcal{L}_\lambda^{\text{tw}}$ into the homogeneous evolution:

$$\frac{d}{dt} \partial O(t) = \mathcal{L}_\lambda^{\text{tw}} \partial O(t) - \frac{1}{2} R \partial O(t) + F(t). \quad (5.4)$$

The creation-gradient equation has $R^\top$. Summing the forcing over sites and both kinds of gradient gives

$$\sum_i (\|F_i(t)\| + \|\bar{F}_i(t)\|) \leq \kappa(\lambda) \mathcal{S}(O(t)).$$

As in (4.8), the two ball-volume factors count affected derivative sites and shell centers containing a fixed gradient site.

The matrix acting on the site index commutes with the superoperator acting on each component. Contractivity of the full twisted semigroup, the column-sum bound of Lemma 3.1, and Duhamel's formula therefore give

$$\mathcal{S}(O(t)) \leq K(t)\mathcal{S}(O) + \kappa(\lambda) \int_0^t K(t-u)\mathcal{S}(O(u)) du. \quad (5.5)$$

The use of the full twisted generator is what avoids an extensive remainder.

Choose $\lambda_*(\beta)$ so that $\kappa(\lambda)A_\beta \leq 1/2$ whenever $|\lambda| \leq \lambda_*(\beta)$. For finite T , let $M_T = \sup_{0 \leq t \leq T} e^{\gamma t} \mathcal{S}(O(t))$. It is finite because the system is finite dimensional. Multiplying (5.5) by $e^{\gamma t}$ and taking this supremum yields

$$M_T \leq B_\beta \mathcal{S}(O) + \kappa(\lambda)A_\beta M_T.$$

Moving the last term to the left and then allowing arbitrary T proves

$$\mathcal{S}(e^{t\mathcal{L}_\lambda} O) \leq 2B_\beta e^{-\gamma t} \mathcal{S}(O). \quad (5.6)$$

6 Mixing, checks, and the range of the result

The Gibbs density is even and fixed by the exact KMS sampler. Evenness of the dynamics allows trace duality for a difference of even states to be restricted to even observables. Subtracting the scalar trace in (5.2) and using (5.6) gives

$$\sup_{\rho \text{ even}} \frac{1}{2} \|e^{t\mathcal{L}_\lambda^*}(\rho) - \rho_{\beta,\lambda}\|_1 \leq 8B_\beta N e^{-\gamma t}. \quad (6.1)$$

For $N \geq 2$ the fixed factor $8B_\beta$ can be absorbed into $C_\beta \log(N/\varepsilon)$. If $N = 1$, every even Hamiltonian is diagonal and the even-state dynamics is a two-state birth–death process. The Gaussian filter is strictly positive on the bounded set of allowed transition energies, so its total rate has a uniform positive lower bound. The same logarithmic form follows directly after increasing C_β . Contraction also proves uniqueness of the even stationary density, completing Theorem 1.1.

The companion evidence contains finite CAR checks, interacting KMS fixed-point checks with the source's exact coherent term, and the strict internal audit. The algebraic controls include complex hopping and nonlinear odd local jumps. They support sign, transpose, and normalization choices; the uniform-in-volume theorem is proved by the spatial kernel, the local continuity estimates, and (5.5).

The constants are allowed to deteriorate with β and the fixed local bounds. No statement uniform in inverse temperature, no global odd-sector uniqueness, and no result for interactions outside the bounded class is inferred. The filter and KMS fixation are inherited from [1]; the proof here develops the physical-site stability argument needed for hopping references. External expert review and a broader novelty comparison remain distinct from the internal checks.

AI-assisted research

This work was developed in an author-directed, AI-assisted research workflow. AI systems assisted with mathematical derivation, literature checks, computational verification, and manuscript drafting. The internal reviews and separate implementations described above were produced within this workflow and do not constitute external expert review.

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