

A certified memory advantage in four-collision non-Gaussian cooling

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Abstract

We compare two cooling protocols for a thermal bosonic target coupled to two thermal machine modes by excitation-converting collisions. Both protocols allow at most four collisions and have the same bound on total interaction time. In one protocol the machines are retained; in the other the selected machine is replaced by a fresh thermal state before every collision. At total time $T = 4/5$, a three-collision retained-machine protocol has final target occupation at most 0.259092738049, whereas every reset protocol has occupation at least 0.259313063246. The proof combines exact invariant sectors, a positive finite subensemble for the reset lower bound, a verified cover of the continuous control simplex, and ball-arithmetic propagation with an analytic infinite-dimensional tail bound. It certifies a strict memory advantage exceeding 0.000220325197.

1 Protocols and the comparison

Let a be a target annihilation operator and b_1, b_2 two machine annihilation operators. We set $\hbar = \omega_S = \omega_M = \beta = 1$ and start all three modes in

$$\tau = (1 - q) \sum_{r=0}^{\infty} q^r |r\rangle \langle r|, \quad q = e^{-1}. \quad (1.1)$$

An allowed collision has the interaction-picture unitary

$$U_j(s, \phi) = \exp \left[-is \left(e^{i\phi} a b_j^{\dagger 2} + e^{-i\phi} a^{\dagger} b_j^2 \right) \right], \quad j \in \{1, 2\}, \quad s \geq 0. \quad (1.2)$$

There are at most four collisions and their durations satisfy $\sum s \leq T$. The objective is the final expectation of $N_S = a^{\dagger} a$.

In the memory protocol, neither machine is reset. In the reset protocol, the selected machine is replaced by an independent copy of τ immediately before each collision. Let $E_{\text{mem}}(T)$ and $E_{\text{reset}}(T)$ be the corresponding infima over the allowed collision lists, durations, and phases. The resource restriction is the same on both sides; neither additional unitaries nor additional interactions are included.

Non-Gaussian excitation conversion can improve bosonic cooling beyond Gaussian operations [1]. The proposal to reuse machines in that setting raises a separate control question: can the correlations retained between collisions improve cooling at a fixed collision count and time budget? Finite-memory improvements in other architectures [2] do not directly answer this constrained comparison.

Theorem 1.1. *For the protocols above at $T = 4/5$,*

$$E_{\text{mem}}(4/5) \leq 0.259092738049, \quad (1.3)$$

$$E_{\text{reset}}(4/5) \geq 0.259313063246. \quad (1.4)$$

In particular,

$$E_{\text{reset}}(4/5) - E_{\text{mem}}(4/5) > 0.000220325197. \quad (1.5)$$

The memory upper bound is attained as a bound on the admissible collision word $(1, 2, 1)$ with phases zero and durations $(217, 404, 179)/1000$.

This is a computer-assisted inequality between the two specified infima. The reset bound covers its entire continuous control domain. The memory bound uses one feasible protocol and makes no claim that this protocol is optimal.

2 Exact reduction of the reset protocol

2.1 Invariant sectors

For a collision with one machine, $K = 2N_S + N_M$ is conserved. In the sector labelled by the nonnegative integer K , use the basis

$$|r, K - 2r\rangle, \quad 0 \leq r \leq \lfloor K/2 \rfloor.$$

At phase zero, the Hamiltonian is the real symmetric tridiagonal matrix H_K with zero diagonal and

$$(H_K)_{r,r+1} = (H_K)_{r+1,r} = \sqrt{(r+1)(K-2r)(K-2r-1)}. \quad (2.1)$$

The direct sum of these finite matrices defines the collision evolution on the Fock space. A phase in (1.2) is removed by a diagonal number rotation, so its transition probabilities in the number basis are unchanged.

Before each reset collision, the target and the fresh selected machine have marginal state $\rho_S \otimes \tau$. This remains true even if the target is correlated with the unused machine. Since the initial target is diagonal and each fixed- K transition leaves a unique machine occupation for each target occupation, tracing out the fresh machine preserves diagonality. Thus every reset protocol is a product of scalar population kernels

$$P_s(r, d) = \sum_{\ell=0}^{\infty} (1-q)q^\ell \left| \left(e^{-isH_{2r+\ell}} \right)_{d,r} \right|^2, \quad (2.2)$$

with unavailable indices interpreted as zero. Neither the machine label nor the phase affects this target kernel. Zero-duration collisions include all protocols with fewer than four collisions.

2.2 A positive subensemble

Retain only initial target occupations $r \leq 10$ and initial occupations $\ell \leq 10$ of each of the four fresh machines. Do not renormalize these weights. The contribution from every omitted initial state to the final occupation is nonnegative; therefore any lower bound for this retained contribution is also a lower bound for the full reset energy.

Define the finite kernel

$$\tilde{P}_s(r, d) = \sum_{\ell=0}^{10} (1-q)q^\ell \left| \left(e^{-isH_{2r+\ell}} \right)_{d,r} \right|^2. \quad (2.3)$$

Its row mass is exactly

$$m_q = 1 - q^{11}. \quad (2.4)$$

In one collision the target occupation can grow by at most five. Before collision $j+1$, $r \leq 10 + 5j$, for $j = 0, 1, 2, 3$; afterwards $r \leq 30$. Consequently rows $0, \dots, 25$, columns $0, \dots, 30$, and sectors $K \leq 60$ contain every retained trajectory. This is a restriction on the initial ensemble, not a projection imposed during the evolution.

For any $s_1, \dots, s_4 \geq 0$ with sum at most $4/5$, the retained energy is

$$E_{\text{sub}}(s_1, \dots, s_4) = \sum_{r=0}^{10} (1-q)q^r \left(\tilde{P}_{s_1} \tilde{P}_{s_2} \tilde{P}_{s_3} \tilde{P}_{s_4} \mathbf{n} \right)_r, \quad \mathbf{n}_d = d. \quad (2.5)$$

It suffices to bound this expression uniformly from below.

3 Certified kernels on time intervals

3.1 A global second-derivative bound

Consider one sector and write $U(s) = e^{-isH_K}$. Since the off-diagonal entries in (2.1) are strictly positive whenever the dimension is larger than one, this Jacobi matrix has simple eigenvalues. Let V be an orthonormal eigenvector matrix with eigenvalues λ_a , and set

$$W_{ra} = |V_{ra}|, \quad A_j = W \operatorname{diag}(\lambda_a^j) W^\top, \quad j = 0, 1, 2.$$

For the transition probability $Q_{dr}(s) = |U(s)_{dr}|^2$,

$$|Q''_{dr}(s)| \leq \sum_{a,b} W_{da} W_{ra} W_{db} W_{rb} (\lambda_a - \lambda_b)^2 \quad (3.1)$$

$$= 2((A_0)_{dr}(A_2)_{dr} - (A_1)_{dr}^2). \quad (3.2)$$

The last equality expands the frequency difference squared. The expression is nonnegative by weighted Cauchy–Schwarz, although A_1 itself may have entries of either sign. This bound holds for every real time. Summing it with the positive thermal weights in (2.3) yields a matrix M_2 satisfying $|\tilde{P}_s''(r, d)| \leq M_2(r, d)$.

The certificate uses verified eigensystem enclosures, normalizes each eigenvector, and encloses (3.2) by Arb arithmetic. Simplicity and Hermiticity identify the enclosed normalized vectors as orthonormal exact eigenvectors. The verifier does not substitute unverified numerical eigenvectors into the formula.

3.2 Center evaluation and outward rounding

For a time interval $I = [c - h, c + h]$, direct ball-arithmetic matrix exponentials give $U(c)$ and $U'(c) = -iH_K U(c)$. Hence

$$Q'_{dr}(c) = 2\Re\left(U'(c)_{dr}\overline{U(c)_{dr}}\right).$$

Taylor's theorem yields entrywise envelopes

$$\tilde{P}_c - h|\tilde{P}'_c| - \frac{1}{2}h^2M_2 \leq \tilde{P}_s \leq \tilde{P}_c + h|\tilde{P}'_c| + \frac{1}{2}h^2M_2, \quad s \in I. \quad (3.3)$$

Every elementary operation, square root, exponential, matrix product, and thermal weight is enclosed at 160-bit Arb/Acb precision using python-flint 0.9.0 with FLINT 3.6.0 [3]. The verified **rump** eigensystem mode is used for the derivative bound; center values use direct matrix exponentials.

Put $S = 10^8$. Round the lower envelope downward and the upper envelope upward onto multiples of S^{-1} , intersecting with nonnegative probabilities and the outward upper row-mass bound. Denote the resulting integer matrices by l_I, u_I . Also set

$$m = \lfloor S(1 - q^{11}) \rfloor, \quad a_r = \lfloor S(1 - q)q^r \rfloor. \quad (3.4)$$

Thus, for every admissible row in the interval,

$$l_{I,rd} \leq S\tilde{P}_s(r, d) \leq u_{I,rd}, \quad \sum_d S\tilde{P}_s(r, d) \geq m.$$

4 A lower bound for every reset schedule

4.1 One row of the backward recursion

Suppose $v_d \geq 0$ is an integer lower bound, in units S^{-1} , on the cost after the next collision. A true row p_d belongs to the relaxation

$$l_d \leq Sp_d \leq u_d, \quad \sum_d Sp_d \geq m.$$

Write $x_d = Sp_d - l_d$ and $\text{need} = m - \sum_d l_d$. For any $\eta \geq 0$,

$$\sum_d p_d \frac{v_d}{S} \geq \frac{1}{S^2} \left[l \cdot v + \eta \text{need} - \sum_d (u_d - l_d) \max\{\eta - v_d, 0\} \right]. \quad (4.1)$$

Indeed, $\sum x_d \geq \text{need}$ and $x_d(v_d - \eta) \geq -(u_d - l_d) \max\{\eta - v_d, 0\}$. Choosing η at the capacity threshold after sorting costs gives the usual exact greedy optimum of this row relaxation. The verifier checks $\text{need} \geq 0$ and that the available capacity is sufficient.

Start from $v_d = Sd$, apply (4.1) backward through four time intervals, and round each resulting value downward to the same cost grid. At the final average, use the downward initial weights a_r/S . All costs are nonnegative, so these roundings preserve the lower bound. Allowing each target occupation to choose its own feasible row within an interval can only lower the common-time optimum. It therefore supplies a valid relaxation of (2.5).

The integer implementation also controls intermediate sizes. The costs obey $0 \leq v_d \leq 30S$. By the choice of threshold, the sum of capacities with $v_d < \eta$ is at most need . Hence the penalty in (4.1) is at most ηneed , while $\sum l_d + \text{need} = m \leq S$. Positive and negative intermediate contributions are bounded by $30S^2 = 3 \times 10^{17}$, below the signed 64-bit limit. The same bound applies to the final initial-state average.

4.2 Covering the continuous simplex

Represent times in ticks of size $(4/5)2^{-18}$. Begin with the closed box $[0, 2^{18}]^4$ in tick coordinates, and bisect the coordinate with largest width. A box with lower endpoint a is discarded only if $\sum_i a_i > 2^{18}$. For a surviving box with upper endpoint b , tighten each upper endpoint to

$$b'_i = \min \left\{ b_i, 2^{18} - \sum_{j \neq i} a_j \right\}. \quad (4.2)$$

This preserves every feasible point in that box. The bisections cover closed intervals, so boundaries and zero-duration protocols remain included.

The saved geometric tree contains 28 097 visited nodes and 14 049 leaves. A separate recursive check verifies that each feasible branch terminates at one of these leaves or an actually infeasible box. It does not trust the original leaf lower bounds. The clipped leaves require 211 distinct one-dimensional time intervals and 177 distinct centers. Recomputing every interval envelope and every backward recursion gives

$$\inf_{\sum_i s_i \leq 4/5} E_{\text{sub}}(s_1, \dots, s_4) \geq \frac{2593130632467262}{10^{16}} = 0.2593130632467262. \quad (4.3)$$

Combining this with the positive-subensemble argument proves the reset part of Theorem 1.1. The tree is a finite certificate of continuous coverage, not a grid search over sample times.

5 The retained-machine upper bound

Use the word $(1, 2, 1)$ with durations

$$s_1 = 217/1000, \quad s_2 = 404/1000, \quad s_3 = 179/1000 \quad (5.1)$$

and phases zero. These add to $4/5$. Each machine's number parity is conserved. Write its occupation as $2u + e_1$ and $2v + e_2$ with $e_1, e_2 \in \{0, 1\}$. Within a fixed parity pair, the quantity

$$L = r + u + v$$

is also conserved. Its sector has basis

$$|r, 2u + e_1, 2v + e_2\rangle, \quad r + u + v = L,$$

and dimension $(L + 1)(L + 2)/2$.

Retain all sectors $L \leq 18$ for all four parity pairs. In each sector propagate the full product $U_1(s_3, 0)U_2(s_2, 0)U_1(s_1, 0)$, using the exact original thermal weight of every initial basis state. No intermediate projection or reset is introduced. For a fixed spectator occupation, each pulse is a block of (2.1), so verified one-machine exponentials assemble the complete evolution. The largest retained sector has dimension 190, and the one-machine blocks have dimension at most 19.

Direct ball-arithmetic propagation gives the retained contribution

$$E_{\leq 18} = 0.25908293553576687947466117711970963664513459 \pm 1.49 \times 10^{-45}. \quad (5.2)$$

The omitted initial mass is small, but a mass estimate alone would not bound the unbounded occupation observable. Instead, let $S_0 = r + \ell_1 + \ell_2$ be the initial total occupation. Conservation of $2N_S + N_{M_1} + N_{M_2}$ implies that final target occupation is at most $r + (\ell_1 + \ell_2)/2 \leq S_0$ on

each initial number-state component. Moreover, $L > 18$ implies $S_0 > 18$. The omitted energy is therefore bounded by

$$\begin{aligned} B_{18} &= \mathbb{E}[S_0 \mathbf{1}_{\{S_0 > 18\}}] \\ &= \frac{3q}{1-q} - (1-q)^3 \sum_{s=0}^{18} s \binom{s+2}{2} q^s \\ &= 9.8025129573887852747006543467614473292495 \times 10^{-6} \pm 8.51 \times 10^{-47}. \end{aligned} \quad (5.3)$$

Here the binomial coefficient counts the triples of nonnegative occupations with sum s .

It follows that

$$E_{\text{mem}}(4/5) \leq E_{\leq 18} + B_{18} < 0.25909273804872426825993587777405639809246385. \quad (5.4)$$

Subtracting (5.4) from (4.3) yields

$$E_{\text{reset}}(4/5) - E_{\text{mem}}(4/5) > 0.0002203251980019317400641.$$

The simpler decimal bounds in Theorem 1.1 follow by outward rounding.

6 Certificate and interpretation

Table 1 summarizes the finite certificate. The companion archive contains the geometric cover, the new Arb verifier, its interval envelopes and output, library-version records, and file hashes. The cover’s SHA-256 digest is recorded in the machine-readable manifest. The verifier reconstructs all leaf energy bounds; previously saved energy values are unused.

Quantity	Certificate value
Working precision	160 bits
Reset sectors	$K = 0, \dots, 60$
Cover nodes / leaves	28 097 / 14 049
Distinct intervals / centers	211 / 177
Probability grid	10^{-8}
Largest center-probability ball radius	6.87×10^{-44}
Retained memory cutoff	$L \leq 18$, all parities
Largest memory sector dimension	190

Table 1: Parameters of the independently implemented arithmetic certificate.

Negative controls verify that zero-time kernels have the expected populations, that a zero-time memory witness cannot establish the claimed gap, and that removing a feasible leaf is detected as a coverage failure. These controls supplement the mathematical derivation; they are not substitutes for interval enclosures or coverage.

The memory protocol retains the machines and their correlations between collisions. The theorem establishes an advantage over mandatory resetting for the precise allowed operations and common budget. It does not isolate coherence as the resource responsible for that advantage, and it does not compare against arbitrary machine control or unbounded collision counts. The new certificate is independent of the earlier implementation’s eigenvector proposals and hand-estimated floating-point budgets. Both implementations were produced within the same research workflow; no external expert verification is claimed.

AI-assisted research

This work was developed in an author-directed, AI-assisted research workflow. AI systems assisted with mathematical derivation, literature checks, computational verification, and manuscript drafting. The internal reviews and separate implementations described above were produced within this workflow and do not constitute external expert review.

References

- [1] W.-H. Png, X. Hu, and V. Scarani, *Sub-Bath Cooling in Bosonic Systems: Gaussian Constraints and Non-Gaussian Enhancements*, PRX Quantum **7**, 020316 (2026). [doi:10.1103/3qnz-6bcz](https://doi.org/10.1103/3qnz-6bcz); [arXiv:2512.10703v2](https://arxiv.org/abs/2512.10703v2).
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- [3] FLINT development team, *FLINT 3.6.0: acb_mat, matrices over the complex numbers*, versioned documentation, including verified eigensystems and matrix exponentials. [Source documentation](#).